ANALYSIS OF POWER AGGREGATION OPERATORS THROUGH CIRCULAR INTUITIONISTIC FUZZY INFORMATION AND THEIR APPLICATIONS IN MACHINE LEARNING ANALYSIS

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ARTICLE INFO

***Article history*:**
Received: 01.08.2024.
Received in revised form: 08.11.2024.
Accepted: 04.12.2024.

Keywords:
Circular intuitionistic fuzzy sets
Power aggregation operators
Machine learning algorithms
Decision-making procedures

DOI: 10.30765/er.2571

Abstract:

Machine learning language is very valuable for depicting different problems, especially computer language, data mining, data sciences, and machine language. The circular intuitionistic fuzzy set (C-IFS) is a flexible approach to fuzzy sets and intuitionistic fuzzy sets. Keep in mind the flexibility of C-IFS, decision-maker used C-IFS to cope with incomplete and redundant human opinions accurately. Furthermore, power operators are used for depicting or aggregating the collection of data into a singleton set. In this manuscript, we explore the power operators for circular intuitionistic fuzzy (C-IF) information, such as C-IF power weighted averaging (C-IFPWA) operator, C-IF power weighted ordered averaging (C-IFPWOA) operator, C-IF power weighted geometric (C-IFPWG) operator, and C-IF power weighted ordered geometric (C-IFPWOG) operator. Some properties of the above information are also stated. Additionally, we evaluate the procedure of the multi-attribute decision-making (MADM) technique for resolving the utilization of the most suitable part of machine learning in complicated scenarios. Finally, we illustrate some numerical examples for addressing the comparison between proposed techniques and existing methods to show the effectiveness and reliability of the presented operators.

1 Introduction

Machine learning algorithms and machine language are two different techniques that are the subpart of artificial intelligence that concentrate on the occurrence or development of algorithms and techniques that enable machines to learn from decisions and predictions. These algorithms are very reliable, and many scholars have employed them in many fields in the consideration of classical set theory. Because of these reasons, experts have faced a lot of problems due to limited opinions. To enhance or modify the range of the decision, Zadeh [1] exposed the fuzzy set theory (FST), where FST has only one function, called truth or positive or membership grade such as: $\mu_A: X \rightarrow [0,1]$, where $\mu_A(x) \in [0,1]$. Some applications of the Zadeh's principle are stated, for instance, extended form of fuzzy sets, called fuzzy superior Mandelbrot set [2], PROMETHEE techniques [3], (a,b)-fuzzy soft sets [4], fuzzy N-soft sets (FNSSs) [5], multi-fuzzy N-soft sets [6], fuzzy parameterized soft sets [7], fuzzy systems [8], multi-person decision-making techniques [9], and fuzzy decision support systems [10].

FST is very strong and effective because FST has only to cope with those problems that cover the one-dimension problems, but in many cases, we face negative information about people, things, and objects. To cope with this kind of situation, Atanassov [11], [12], [13] explored the intuitionistic FST (IFST), where IFST

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talked about the positive and negative grades with the characteristic that the sum of the pair will be contained in a unit interval. IFST has two different grades with the same domain and range, where the FST and classical set theory are the special cases of the IFST. Many scholars have utilized the IFST in many fields, for instance, similarity measures with application in emergency management and pattern recognition [14], 3D distance measures and their application in decision-making problems [15], analysis of multi-objective decision-making techniques [16], generalized similarity operators [17], distance and similarity measures [18], analysis and classification of parametric divergence measures [19], fairly operators and additive ratio assessments [20], and analysis of time-series based on higher order with IFST [21].

Most scholars accept the structure of IFST because it contains positive and negative grades, but during the analysis of rain, we have three possibilities, for instance, to give his opinion in favor of rain, to give his opinion against rain, and one of the most opinion, called the angle of the rain, because the angle is very important. After all, if we know the angle of the rain before starting, we will save ourselves. Therefore, to handle such kind of problems, the circular-IFS (C-IFS) was invented by Atanassov [22]. Many applications have been discussed as follows, for instance, analysis of four distance measures [23], divergence measures [24], TOPSIS techniques [25], interval-valued C-IFSs [26] TOPSIS method [27], decision-making approaches [28], AHP techniques [29], involved distance measures [30], AHP techniques means that analytical hierarchy process [31], and advanced approach and decision-making techniques for C-IFSs [32].

Aggregating the collection of information into a singleton set is very complicated and vague because it is a very challenging task for scholars. Additionally, Yager [33] also invented the power aggregation operators for classical set theory, which is a suitable and dominant technique for depicting awkward and vague data. Furthermore, the simple average operator (AVO) for IFST was exposed by Xu [34]. Moreover, the geometric operator (GEO) for IFST was exposed by Xu and Yager [35]. In 2018, Jiang et al. [36] derived the power AVO and power GEO for IFST. Hussain et al. [37] designed innovative approaches for Aczel Alsina operators for handling uncertain information of human opinions. Hussain et al. [38] demonstrated the characteristics of different solar panels to investigate the best optimal option under considering different features. Hussain et al. [39] modified the theory of complex picture fuzzy information to select a suitable supplier with decision analysis processes. Hussain et al. [40] put forward the concepts of Hernonian mean operators using Aczel Alsina operations. Hussain et al. [41] presented a robust selection process to evaluate different recycling techniques using Dombi Bonferroni Mean operators and a decision analysis process. Hussain et al. [42] proposed AOs of Sugeno-Weber t-norms considering an intuitionistic fuzzy system. Hussain et al. [43] developed an intelligent decision-making model using Frank AOs and complex picture fuzzy theory. Wang et al. [44] presented mathematical terminologies of Sugeno-Weber t-norms based on q-rung orthopair fuzzy domains. Hussain et al. [45] utilized various properties of Hamy mean models to define correction among input data. Ali et al. [46] enhanced various characteristics of Fermatean fuzzy theory and deduced new algebraic AOs for aggregating human opinions. Abed Alhaleem and Ahmad [47] demonstrated new approaches to intuitionistic fuzzy domains.

Uluçay and Okumuş [48] enhanced the dealing capacity of an intuitionistic trapezoidal fuzzy theory and also investigated a sustainable tourism industry. Imran et al. [49] designed some robust mathematical approaches to Aczel Alsina AOs and Bonferroni Mean models. Sahoo et al. [50] introduced a robust binary-coded genetic algorithm to investigate suitable supply chain enterprises. Asif et al. [51] constructed AOs of Hamacher t-norms under considering the theory of pythagorean fuzzy environment. Mishra et al. [52] applied a novel approach of an interval-valued intuitionistic fuzzy domain and distance measures to examine sustainable wastewater sources. Hussain and Ullah [53] put forward the concept of an advanced decision analysis process and Sugeno-Weber mathematical approaches. Ahmmad [54] classified some reliable energy sources using properties of entropy measures and q-rung orthopair fuzzy situations. Ali [55] discussed innovative approaches to power interaction AOs under consideration complex IFSs. Mahmood et al. [56] deduced robust mathematical approaches and decision analysis processes to resolve an application related to medical diagnosis.

Hussain and Pamucar [57] constructed an intelligent decision-making model and AOs of Schweizer-Sklar t-norms based on pythagorean fuzzy information. Ahn et al. [58] applied the theory of an interval-valued IFS to find an authentic solution for medical diagnosis. Bibi and Ali [59] designed a dominant structure of Aczel Alsina AOs and decision-making methodologies. Hussain et al. [60] developed Dombi AOs based on the interval-valued spherical fuzzy framework.

1.1 Motivation Behind the Research Work

The CIFSs are an advancement in FS theory designed to enhance flexibility and precision in handling uncertainty. Traditional FSs allow for degrees of membership and non-membership to represent uncertainty, while IFSs add a third parameter, the hesitation margin, to address instances where there is insufficient information. C-IFSs, however, go a step further by integrating a circular representation of membership, non-membership, and hesitation, offering a more nuanced and visual representation of these values. This circular model allows better representation of complex data structures where relationships are non-linear or cyclical. The primary motivation for C-IFSs is to improve decision-making processes, especially in fields where data ambiguity and cyclical relationships are common, such as economics, environmental sciences, and social sciences. By allowing analysts to capture more detailed uncertainties, C-IFSs can lead to more accurate and context-sensitive decision outcomes.

Power aggregation operators are developed to address limitations in traditional aggregation methods, particularly when dealing with highly diverse data or data that includes outliers. Traditional operators, like arithmetic means or weighted averages, may underperform or lead to biased results in cases where some values in the dataset significantly differ from others. Power aggregation operators mitigate this by incorporating exponential functions, allowing for more control over how individual data points influence the overall aggregation. These operators are particularly useful in applications like risk assessment and financial analysis, where extreme values or non-linear relationships are prevalent. The motivation behind power aggregation operators is to enhance the robustness and adaptability of aggregation methods, making them better suited for decision-making under uncertainty. By adjusting the power parameter, these operators can emphasize or downplay specific values, resulting in more reliable outcomes and allowing for fine-tuned analysis.

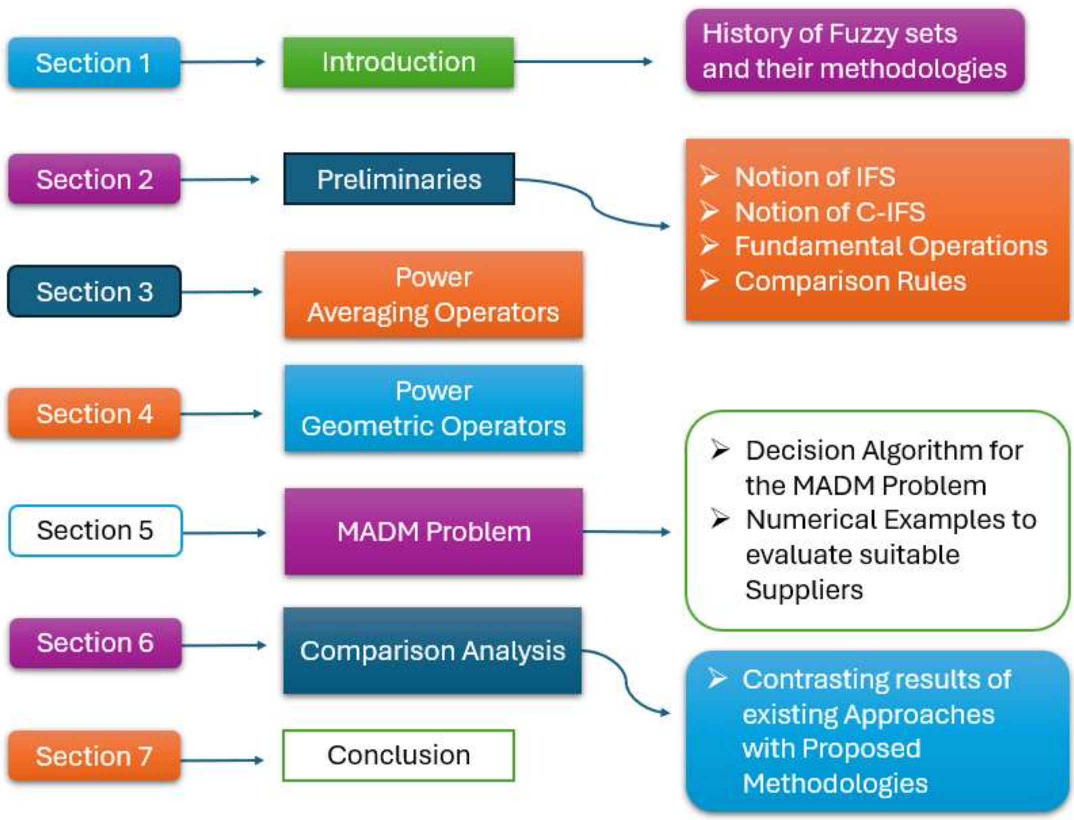

Figure 1. Shows the section-wise structure of the manuscript.

Decision-making approaches for machine learning operators are motivated by the need for reliable, interpretable, and effective ways to implement ML models in real-world scenarios. Machine learning models often operate as "black boxes," which limits transparency and interpretability. This is especially challenging in high-stakes decision environments, like healthcare, finance, and autonomous driving, where the rationale behind a model's output is crucial. Decision-making approaches are designed to address this by providing frameworks or operators that enhance the model's interpretability and explainability, improving trust and accountability. Furthermore, decision-making approaches tailored for ML can also optimize model selection, tuning, and integration with broader decision systems. In addition, these approaches facilitate alignment with ethical standards and regulatory compliance, making machine-learning applications more suitable for use in sensitive or high-impact fields. By grounding machine learning operations in systematic decision-making methods, these approaches aim to make the use of ML models both safer and more effective in supporting complex human decisions.

In the above discussion, we noticed that the theory of C-IFS is novel, and no one can derive a lot of operators based on C-IFS, the selected operators, called power operators, were not invented by anyone. Therefore, our main contribution is listed below:

1) Expose an innovative theory of C-IFSs for handling uncertain information of expert opinions.
2) To explore the C-IFPWA operator, C-IFPWOA operator, C-IFPWG operator, and C-IFPWOG operator.
3) To derive the three basic properties of the above information.
4) To evaluate the procedure of the MADM technique for resolving the utilization of the most suitable part of machine learning in complicated scenarios.
5) To illustrate some numerical examples for addressing the comparison between proposed techniques and existing methods to show the effectiveness and reliability of the presented operators.

This manuscript is arranged in shape: In Section 2, we revised the idea of PA operator, PG operator, C-IFS, and their operational laws. In Section 3, we explored the C-IFPWA operator, C-IFPWOA operator. In section 4, we also constructed a series of geometric AOs such as C-IFPWG operator, and C-IFPWOG operator. Some properties of the above information are also stated. In Section 5, we evaluated the procedure of the MADM technique for resolving the utilization of the most suitable part of machine learning in complicated scenarios. In Section 6, we illustrated some numerical examples for addressing the comparison between proposed techniques and existing methods to show the effectiveness and reliability of the presented operators. Some concluding remarks are stated in Section 7. Figure 1 depicts the section-wise organization of this article.

2 Preliminaries

In this section, we revised the idea of PA operator, PG operator, C-IFS, and their operational laws.

Definition 1: [33] For any finite family of positive integers $a_i, (i = 1,2,\dots,n)$. The PA operator is invented by (1):

$$PA\,(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} \frac{(1+\mathcal{A}(\alpha_i))}{\sum_{i=1}^{n}(1+\mathcal{A}(\alpha_i))} \alpha_i \tag{1}$$

Noticed that Where $\frac{(1+\mathcal{A}(\alpha_i))}{\sum_{i=1}^{n}(1+\mathcal{A}(\alpha_i))}$ and $\mathcal{A}(\alpha_i) = \sum_{\substack{i=1 \\ j\neq i}}^{n} Supp(\alpha_i, \alpha_j), (i = 1, 2, \dots, n)$ represents the support degree between a_i and a_j, with some properties, such as:

a) $Sup(a_i, a_j) \in [0,1]$

b) $Sup(a_i, a_j) = Sup(a_j, a_i)$

c) $Sup(a_i, a_j) \geq Sup(a_s, a_t)$, if $|a_i, a_j| < |a_s, a_t|$

Definition 2: [33] For any finite family of positive integers $a_i (i = 1,2,\dots,n)$. The PG operator is invented by (2):

$$PG(a_1, a_2, \dots, a_n) = \prod_{i=1}^{n} a_i^{\frac{(1+\mathcal{A}(\alpha_i))}{\sum_{i=1}^{n}(1+\mathcal{A}(\alpha_i))}} \tag{2}$$

Noticed that Where $\frac{(1+\mathcal{A}(\alpha_i))}{\sum_{i=1}^{n}(1+\mathcal{A}(\alpha_i))}$ and $\mathcal{A}(\alpha_i) = \sum_{\substack{i=1 \\ j\neq i}}^{n} Sup(\alpha_i, \alpha_j), (i = 1, 2, \dots, n)$ represents the support degree between a_i and a_j, with some properties, such as:

a) $Sup(a_i, a_j) \in [0,1]$

b) $Sup(a_i, a_j) = Sup(a_j, a_i)$

c) $Sup(a_i, a_j) \geq Sup(a_s, a_t)$, if $|a_i, a_j| < |a_s, a_t|$

Definition 3: [11] For the universal set E, an IFS is expressed as follows (3):

$$A = \{(x, \mu_A(x), v_A(x)) | x \in E\} \quad (3)$$

Noticed that $\mu_A(x) \in [0,1]$ and $v_A(x) \in [0,1]$ denote the positive grade and negative grade respectively with subject to condition: $0 \leq \mu_A(x) + v_A(x) \leq 1$. Additionally, $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ is denoted the hesitancy value of A. Further, a pair $(\mu_A(x), v_A(x))$ represents an intuitionistic fuzzy value.
Definition 4: [22] For the universal set E, the C-IFS is invented by (4):

$$A = \{< x, \mu_A(x), v_A(x);\ r_A > | x \in E\} \quad (4)$$

Noticed that $\mu_A(x)$, denotes the positive grades and $v_A(x)$, denotes the negative grades with $\mu_A(x), v_A(x) \geq 0$ and $0 \leq \mu_A(x) + v_A(x) \leq 1,\ \forall x \in E$, where r represents the radius of the point $(\mu_A(x), v_A(x))$. Additionally, $\pi_A(x) = 1 - \mu_A(x), -v_A(x)$ represents the hesitancy value of C-IFS and a triplet $\mathfrak{W} = \left(\mu_{\mathfrak{W}_i}(x), v_{\mathfrak{W}_i}(x);\ r_{\mathfrak{W}_i}(x)\right)$ is known as the circular intuitionistic fuzzy value (C-IFV).
Definition 5: [61] The mathematical shape of score value $s_{\mathfrak{W}}$ and accuracy value $h_{\mathfrak{W}}$ is invented by (5,6):

$$s_{\mathfrak{W}} = (\mu_{\mathfrak{W}} - v_{\mathfrak{W}}), s_{\mathfrak{W}} \in [-1,1] \quad (5)$$

$$h_{\mathfrak{W}} = (\mu_{\mathfrak{W}} + v_{\mathfrak{W}}), h_{\mathfrak{W}} \in [0,1] \quad (6)$$

For simplification, we have some rules:

- If $s_{\mathfrak{W}_1} > s_{\mathfrak{W}_2}$, then $\mathfrak{W}_1 > \mathfrak{W}_2$
- If $s_{\mathfrak{W}_1} = s_{\mathfrak{W}_2}$, then:
  1) If $h_{\mathfrak{W}_1} = h_{\mathfrak{W}_2}$, then $\mathfrak{W}_1 = \mathfrak{W}_2$
  2) If $h_{\mathfrak{W}_1} > h_{\mathfrak{W}_2}$, then $\mathfrak{W}_1 > \mathfrak{W}_2$

Definition 6: [61] Consider any two C-IFVs, $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}\right)(i = 1,2)$, Then, some flexible operations for C-IFVs are discussed as follows (7-14):

$$\mathfrak{W}_1 \oplus_t \mathfrak{W}_2 = (\mu_{\mathfrak{W}_1} + \mu_{\mathfrak{W}_2} - \mu_{\mathfrak{W}_1}\mu_{\mathfrak{W}_2},\ v_{\mathfrak{W}_1}v_{\mathfrak{W}_2}, r_{\mathfrak{W}_1} + r_{\mathfrak{W}_2} - r_{\mathfrak{W}_1}r_{\mathfrak{W}_2}) \quad (7)$$

$$\mathfrak{W}_1 \oplus_{tc} \mathfrak{W}_2 = (\mu_{\mathfrak{W}_1} + \mu_{\mathfrak{W}_2} - \mu_{\mathfrak{W}_1}\mu_{\mathfrak{W}_2},\ v_{\mathfrak{W}_1}v_{\mathfrak{W}_2}, r_{\mathfrak{W}_1}r_{\mathfrak{W}_2}) \quad (8)$$

$$\mathfrak{W}_1 \otimes_t \mathfrak{W}_2 = (\mu_{\mathfrak{W}_1}\mu_{\mathfrak{W}_2}, v_{\mathfrak{W}_1} + v_{\mathfrak{W}_2} - v_{\mathfrak{W}_1}v_{\mathfrak{W}_2}, r_{\mathfrak{W}_1} + r_{\mathfrak{W}_2} - r_{\mathfrak{W}_1}r_{\mathfrak{W}_2}) \quad (9)$$

$$\mathfrak{W}_1 \otimes_{tc} \mathfrak{W}_2 = (\mu_{\mathfrak{W}_1}\mu_{\mathfrak{W}_2}, v_{\mathfrak{W}_1} + v_{\mathfrak{W}_2} - v_{\mathfrak{W}_1}v_{\mathfrak{W}_2}, r_{\mathfrak{W}_1}r_{\mathfrak{W}_2}) \quad (10)$$

$$\varphi\mathfrak{W}_{1_t} = (1-(1-\mu_{\mathfrak{W}_1})^{\lambda}, v_{\mathfrak{W}_1}^{\lambda}, 1-(1-r_{\mathfrak{W}_1})^{\lambda}), \lambda > 0 \tag{11}$$

$$\varphi\mathfrak{W}_{1_{tc}} = (1-(1-\mu_{\mathfrak{W}_1})^{\lambda}, v_{\mathfrak{W}_1}^{\lambda}, r_{\mathfrak{W}_1}^{\lambda}), \lambda > 0 \tag{12}$$

$$\mathfrak{W}_{1_t}^{\lambda} = (\mu_{\mathfrak{W}_1}^{\lambda}, 1-(1-v_{\mathfrak{W}_1})^{\lambda}, r_{\mathfrak{W}_1}^{\lambda}), \lambda > 0 \tag{13}$$

$$\mathfrak{W}_{1_{tc}}^{\lambda} = (\mu_{\mathfrak{W}_1}^{\lambda}, 1-(1-v_{\mathfrak{W}_1})^{\lambda}, 1-(1-r_{\mathfrak{W}_1})^{\lambda}), \lambda > 0.0 \tag{14}$$

3 Power Aggregation Operators Based on C-IFSs

This section includes the C-IFPWA, C-IFPWG, C-IFPOWA, and C-IFPWOG operators. We also derive some basic properties and special cases.

Definition 7: Consider $\mathfrak{W}_i = (\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i}; r_{\mathfrak{W}_i}), (i = 1, 2, \dots, n)$ be the collection of C-IFVs and the C-IFPWA operators are defined as follows (15,16):

$$C-IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t = \bigoplus_{i=1}^{n} w_i \alpha_i \tag{15}$$

$$C-IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} = \bigoplus_{i=1}^{n} w_i \alpha_i \tag{16}$$

Where $w_i = \frac{\mathfrak{w}_i(1+\mathcal{A}(\alpha_i))}{\sum_{i=1}^{n}\mathfrak{w}_i(1+\mathcal{A}(\alpha_i))}, \mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_n)$ be the set of weights and $\mathcal{A}(\alpha_i) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \mathfrak{w}_i Sup(\alpha_i, \alpha_j), (i = 1, 2, \dots, n)$.

Theorem 1: To consider the information in definition 7, it can be shown that the aggregated value is also a C-IFV, as (17,18):

$$C-IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t = \begin{pmatrix} 1-\prod_{i=1}^{n}(1-\mu_i)^{w_i}, \\ \prod_{i=1}^{n}(v_i)^{w_i}, \\ 1-\prod_{i=1}^{n}(1-r_i)^{w_i} \end{pmatrix} \tag{18}$$

$$C-IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} = \begin{pmatrix} 1-\prod_{i=1}^{n}(1-\mu_i)^{w_i}, \\ \prod_{i=1}^{n}(v_i)^{w_i}, \\ \prod_{i=1}^{n}(r_i)^{w_i} \end{pmatrix} \tag{19}$$

Proof: Since $\mathfrak{W}_i = (\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i}; r_{\mathfrak{W}_i}), (i = 1, 2, \dots, n)$ be a set of C-IFVs and we prove the above expression for $n = 2$ (20-24):

$$C-IFPWA(\mathfrak{W}_1, \mathfrak{W}_2)_t = \bigoplus_{i=1}^{2} w_i \alpha_i \tag{20}$$

$$w_1\mathfrak{W}_1 = (1-(1-\mu_1)^{w_1}, v_1^{w_1}, 1-(1-r_1)^{w_1}) \tag{21}$$

$$w_2\mathfrak{W}_2 = (1-(1-\mu_2)^{w_2}, v_2^{w_2}, 1-(1-r_2)^{w_2}) \tag{22}$$

$$C-IFPWA(\mathfrak{W}_1, \mathfrak{W}_2)_t = \bigoplus_{i=1}^{2} w_i \alpha_i \tag{23}$$

$$w_1\mathfrak{W}_1 \oplus_t w_2\mathfrak{W}_2 = \left(1-\prod_{i=1}^{2}(1-\mu_i)^{w_i}, \prod_{i=1}^{2}(v_i)^{w_i}, 1-\prod_{i=1}^{2}(1-r_i)^{w_i}\right) \tag{24}$$

Suppose that the above expression is true for $n = k$ and we have (25):

$$C - IFPWA(\mathfrak{W}_1, \mathfrak{W}_2)_t = \bigoplus_{i=1}^{k} \mathcal{w}_i \alpha_i$$

$$= \left(1 - \prod_{i=1}^{k}(1-\mu_i)^{\mathcal{w}_i}, \prod_{i=1}^{k}(v_i)^{\mathcal{w}_i}, 1 - \prod_{i=1}^{k}(1-r_i)^{\mathcal{w}_i}\right) \tag{25}$$

Next, we have to prove $n = k + 1$ (26):

$$C - IFPWA(\mathfrak{W}_1, \mathfrak{W}_2)_t = \bigoplus_{i=1}^{k} \mathcal{w}_i \alpha_i \oplus \mathcal{w}_{k+1} \alpha_{k+1}$$

$$= \begin{pmatrix} 1 - \prod_{i=1}^{k}(1-\mu_i)^{\mathcal{w}_i}, \prod_{i=1}^{k}(v_i)^{\mathcal{w}_i}, \\ 1 - \prod_{i=1}^{k}(1-r_i)^{\mathcal{w}_i} \end{pmatrix} \oplus \begin{pmatrix} 1 - (1-\mu_{k+1})^{\mathcal{w}_{k+1}}, (v_{k+1})^{\mathcal{w}_{k+1}}, \\ 1 - (1-r_{k+1})^{\mathcal{w}_{k+1}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \prod_{i=1}^{k+1}(1-\mu_i)^{\mathcal{w}_i}, \prod_{i=1}^{k+1}(v_i)^{\mathcal{w}_i}, \\ 1 - \prod_{i=1}^{k+1}(1-r_i)^{\mathcal{w}_i} \end{pmatrix} \tag{26}$$

We can also prove the remaining proof using stepwise expressions of the above proof.
Property 1: Let $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}\right), (i = 1, 2, \dots, n)$ be a set of C-IFVs, if $\alpha_i = \alpha$, for all j, then (27,28):

$$C - IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t = \alpha \tag{27}$$

$$C - IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} = \alpha \tag{28}$$

Proof: We can prove the idempotency property as follows (29):

$$C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t = \begin{pmatrix} 1 - \prod_{i=1}^{n}(1-\mu_{\Theta(i)})^{\mathcal{w}_i}, \\ \prod_{i=1}^{n}(v_{\Theta(i)})^{\mathcal{w}_i}, \\ 1 - \prod_{i=1}^{n}(1-r_{\Theta(i)})^{\mathcal{w}_i} \end{pmatrix} \tag{29}$$

Since each C-IFV is identical as $\alpha_i = \alpha$, so we have (30):

$$C - IFPWG(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t = \begin{pmatrix} \prod_{i=1}^{n}(\mu_i)^{\mathcal{w}_i}, \\ 1 - \prod_{i=1}^{n}(1-v_i)^{\mathcal{w}_i}, \\ \prod_{i=1}^{n}(r_i)^{\mathcal{w}_i} \end{pmatrix}$$

$$= \begin{pmatrix} (\mu)^{\sum_{i=1}^{n}\mathcal{w}_i}, \\ 1 - (1-v)^{\sum_{i=1}^{n}\mathcal{w}_i}, \\ (r)^{\sum_{i=1}^{n}\mathcal{w}_i} \end{pmatrix}, \sum_{i=1}^{n} \mathcal{w}_i = 1$$

$$= (\mu, v, r) \tag{30}$$

Property 2: Let $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i}; r_{\mathfrak{W}_i}\right), (i = 1, 2, \dots, n)$ is a collection of n C-IFVs, and $(\alpha_1', \alpha_2', \dots, \alpha_n')$ be any permutation of $(\alpha_1, \alpha_2, \dots, \alpha_n)$, then (31-32):

$$C - IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t \le C - IFPWA(\mathfrak{W}_1', \mathfrak{W}_2', \dots, \mathfrak{W}_n')_t \tag{31}$$

$$C - IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} \le C - IFPWA(\mathfrak{W}_1', \mathfrak{W}_2', \dots, \mathfrak{W}_n')_{tc} \tag{32}$$

Proof: we can easily prove the above expressions.
Property 3: Let $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}\right), (i = 1, 2, \dots, n)$ be a set of C-IFVs, then (33-34):

$$\alpha_{min} \le C - IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t \le \alpha_{max} \tag{33}$$

$$\alpha_{min} \le C - IFPWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} \le \alpha_{max} \tag{34}$$

Where $\alpha_{min} = \left(min_i\{\mu_{\alpha_i}\}, max_i\{v_{\alpha_i}\}\right)$ and $\alpha_{max} = \left(max_i\{\mu_{\alpha_i}\}, min_i\{v_{\alpha_i}\}\right)$.

Proof: is analogous.

Definition 8: Consider $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i}; r_{\mathfrak{W}_i}\right), (i = 1, 2, \dots, n)$ be the collection of C-IFVs and the C-IFPWA operators are defined as follows (35-36):

$$C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t = \bigoplus_{i=1}^{n} \mathcal{w}_i \alpha_{\Theta(i)} \tag{35}$$

$$C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} = \bigoplus_{i=1}^{n} \mathcal{w}_i \alpha_{\Theta(i)} \tag{36}$$

Where $\mathcal{w}_i = \frac{\mathfrak{w}_i(1+\mathcal{A}(\alpha_i))}{\sum_{i=1}^{n} \mathfrak{w}_i(1+\mathcal{A}(\alpha_i))}$, $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_n)$ be the set of weights and $\mathcal{A}(\alpha_i) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \mathfrak{w}_i Sup\left(\alpha_i, \alpha_j\right), (i = 1, 2, \dots, n)$. Furthermore, $\left(\Theta(1), \Theta(2), \dots, \Theta(n)\right)$ be the set of permutations of $\mathfrak{W}_i$ such as $\Theta(i) \le \Theta(i+1)$.

Theorem 2: To consider the information in definition 8, it can be shown that the aggregated value is also a C-IFV, as (37-38):

$$C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t = \begin{pmatrix} 1 - \prod_{i=1}^{n}(1 - \mu_{\Theta(i)})^{\mathcal{w}_i}, \\ \prod_{i=1}^{n}(v_{\Theta(i)})^{\mathcal{w}_i}, \\ 1 - \prod_{i=1}^{n}(1 - r_{\Theta(i)})^{\mathcal{w}_i} \end{pmatrix} \tag{37}$$

$$C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} = \begin{pmatrix} 1 - \prod_{i=1}^{n}(1 - \mu_{\Theta(i)})^{\mathcal{w}_i}, \\ \prod_{i=1}^{n}(v_{\Theta(i)})^{\mathcal{w}_i}, \\ \prod_{i=1}^{n}(r_{\Theta(i)})^{\mathcal{w}_i} \end{pmatrix} \tag{38}$$

Property 4: Let $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i}; r_{\mathfrak{W}_i}\right), (i = 1, 2, \dots, n)$ is a set of C-IFVs, if $\alpha_i = \alpha$, for all j, then (39-40):

$$C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t = \alpha \tag{39}$$

$$C - IFPOWA = \alpha \tag{40}$$

Property 5: Let $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i}; r_{\mathfrak{W}_i}\right), (i = 1, 2, \dots, n)$ is a vector of n C-IFVs, and $(\alpha_1', \alpha_2', \dots, \alpha_n')$ be any permutation of $(\alpha_1, \alpha_2, \dots, \alpha_n)$, then (41-42):

$$C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t \le C - IFPOWA(\mathfrak{W}_1', \mathfrak{W}_2', \dots, \mathfrak{W}_n')_t \tag{41}$$

$$C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} \le C - IFPOWA(\mathfrak{W}_1', \mathfrak{W}_2', \dots, \mathfrak{W}_n')_{tc} \tag{42}$$

Property 6: Let $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i}; r_{\mathfrak{W}_i}\right), (i = 1, 2, \dots, n)$ is the set of C-IFVs, then (42-43):

$$\alpha_{min} \le C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_t \le \alpha_{max} \tag{42}$$

$$\alpha_{min} \le C - IFPOWA(\mathfrak{W}_1, \mathfrak{W}_2, \dots, \mathfrak{W}_n)_{tc} \le \alpha_{max} \tag{43}$$

Where $\alpha_{min} = \left(min_i\{\mu_{\alpha_i}\}, max_i\{v_{\alpha_i}\}\right)$ and $\alpha_{max} = \left(max_i\{\mu_{\alpha_i}\}, min_i\{v_{\alpha_i}\}\right)$.

4 Power Geometric Aggregation Operators Based on C-IFSs

In this section, we constructed a series of geometric aggregation operators for C-IFSs.

Definition 9: Consider $\mathfrak{W}_i = \left(\mu_{\mathfrak{W}_i}, v_{\mathfrak{W}_i}; r_{\mathfrak{W}_i}\right), (i = 1, 2, \dots, n)$ be the collection of C-IFVs and the C-IFPWG operators are defined as follows (44-45):

$$C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t=\underset{i=1}{\overset{n}{\otimes}}\alpha_i^{\mathcal{w}_i} \tag{45}$$

$$C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}=\underset{i=1}{\overset{n}{\otimes}}\alpha_i^{\mathcal{w}_i} \tag{45}$$

Where $\mathcal{w}_i=\frac{\mathfrak{w}_i(1+\mathcal{A}(\alpha_i))}{\sum_{i=1}^n\mathfrak{w}_i(1+\mathcal{A}(\alpha_i))}$, $\mathfrak{w}=(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_n)$ be the set of weights and $\mathcal{A}(\alpha_i)=\sum_{\substack{i=1\\i\neq j}}^n\mathfrak{w}_iSup(\alpha_i,\alpha_j)$, $(i=1,2,\dots,n)$.

Theorem 3: To consider the information in definition 7, it can be shown that the aggregated value is also a C-IFV, as (46-47):

$$C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t=\begin{pmatrix}\prod_{i=1}^n(\mu_i)^{\mathcal{w}_i},\\1-\prod_{i=1}^n(1-v_i)^{\mathcal{w}_i},\\\prod_{i=1}^n(r_i)^{\mathcal{w}_i}\end{pmatrix} \tag{46}$$

$$C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}=\begin{pmatrix}\prod_{i=1}^n(\mu_i)^{\mathcal{w}_i},\\1-\prod_{i=1}^n(1-v_i)^{\mathcal{w}_i},\\1-\prod_{i=1}^n(1-r_i)^{\mathcal{w}_i}\end{pmatrix} \tag{47}$$

Property 7: Let $\mathfrak{W}_i=(\mu_{\mathfrak{W}_i},v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}),(i=1,2,\dots,n)$ is a set of C-IFVs, if $\alpha_i=\alpha$, for all j, then (48-49):

$$C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t=\alpha \tag{48}$$

$$C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}=\alpha \tag{49}$$

Property 8: Let $\mathfrak{W}_i=(\mu_{\mathfrak{W}_i},v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}),(i=1,2,\dots,n)$ is a vector of n C-IFVs, and $(\alpha_1',\alpha_2',\dots,\alpha_n')$ be any permutation of $(\alpha_1,\alpha_2,\dots,\alpha_n)$, then (50-51):

$$C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t\le C-IFPWG(\mathfrak{W}_1',\mathfrak{W}_2',\dots,\mathfrak{W}_n')_t \tag{50}$$

$$C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}\le C-IFPWG(\mathfrak{W}_1',\mathfrak{W}_2',\dots,\mathfrak{W}_n')_{tc} \tag{51}$$

Property 9: Let $\mathfrak{W}_i=(\mu_{\mathfrak{W}_i},v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}),(i=1,2,\dots,n)$ is the set of C-IFVs, then (52-53):

$$\alpha_{min}\le C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t\le\alpha_{max} \tag{52}$$

$$\alpha_{min}\le C-IFPWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}\le\alpha_{max} \tag{53}$$

Where $\alpha_{min}=(min_i\{\mu_{\alpha_i}\},max_i\{v_{\alpha_i}\})$ and $\alpha_{max}=(max_i\{\mu_{\alpha_i}\},min_i\{v_{\alpha_i}\})$.

Definition 10: Consider $\mathfrak{W}_i=(\mu_{\mathfrak{W}_i},v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}),(i=1,2,\dots,n)$ be the collection of C-IFVs and the C-IFPWG operators are defined as follows (54-55):

$$C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t=\underset{i=1}{\overset{n}{\otimes}}\alpha_{\Theta(i)}^{\mathcal{w}_i} \tag{54}$$

$$C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}=\underset{i=1}{\overset{n}{\otimes}}\alpha_{\Theta(i)}^{\mathcal{w}_i} \tag{55}$$

Where $\mathcal{w}_i=\frac{\mathfrak{w}_i(1+\mathcal{A}(\alpha_i))}{\sum_{i=1}^n\mathfrak{w}_i(1+\mathcal{A}(\alpha_i))}$, $\mathfrak{w}=(\mathfrak{w}_1,\mathfrak{w}_2,\dots,\mathfrak{w}_n)$ be the set of weights and $\mathcal{A}(\alpha_i)=\sum_{\substack{i=1\\i\neq j}}^n\mathfrak{w}_iSup(\alpha_i,\alpha_j)$, $(i=1,2,\dots,n)$. Furthermore, $(\Theta(1),\Theta(2),\dots,\Theta(n))$ be the set of permutations of $\mathfrak{W}_i$ such as $\Theta(i)\le\Theta(i+1)$.

Theorem 4: To consider the information in definition 7, it can be shown that the aggregated value is also a C-IFV, as (56-57):

$$C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t=\begin{pmatrix}\prod_{i=1}^{n}(\mu_{\Theta(i)})^{w_i},\\ 1-\prod_{i=1}^{n}(1-v_{\Theta(i)})^{w_i},\\ \prod_{i=1}^{n}(r_{\Theta(i)})^{w_i}\end{pmatrix} \tag{56}$$

$$C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}=\begin{pmatrix}\prod_{i=1}^{n}(\mu_{\Theta(i)})^{w_i},\\ 1-\prod_{i=1}^{n}(1-v_{\Theta(i)})^{w_i},\\ 1-\prod_{i=1}^{n}(1-r_{\Theta(i)})^{w_i}\end{pmatrix} \tag{57}$$

Property 10: Let $\mathfrak{W}_i=\left(\mu_{\mathfrak{W}_i},v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}\right),(i=1,2,\dots,n)$ is a set of C-IFVs, if $\alpha_i=\alpha$, for all j, then (58-59):

$$C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t=\ \alpha \tag{58}$$

$$C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}=\ \alpha \tag{59}$$

Property 11: Let $\mathfrak{W}_i=\left(\mu_{\mathfrak{W}_i},v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}\right),(i=1,2,\dots,n)$ is a vector of n C-IFVs, and $(\alpha_1',\alpha_2',\dots,\alpha_n')$ be any permutation of $(\alpha_1,\alpha_2,\dots,\alpha_n)$, then (60-61):

$$C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t\leq C-IFPOWG(\mathfrak{W}_1',\mathfrak{W}_2',\dots,\mathfrak{W}_n')_t \tag{60}$$

$$C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}\leq C-IFPOWG(\mathfrak{W}_1',\mathfrak{W}_2',\dots,\mathfrak{W}_n')_{tc} \tag{61}$$

Property 12: Let $\mathfrak{W}_i=\left(\mu_{\mathfrak{W}_i},v_{\mathfrak{W}_i};\ r_{\mathfrak{W}_i}\right),(i=1,2,\dots,n)$ is the set of C-IFVs, then (62-63):

$$\alpha_{min}\leq C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_t\leq\alpha_{max} \tag{62}$$

$$\alpha_{min}\leq C-IFPOWG(\mathfrak{W}_1,\mathfrak{W}_2,\dots,\mathfrak{W}_n)_{tc}\leq\alpha_{max} \tag{63}$$

Where $\alpha_{min}=\left(min_i\{\mu_{\alpha_i}\},max_i\{v_{\alpha_i}\}\right)$ and $\alpha_{max}=\left(max_i\{\mu_{\alpha_i}\},min_i\{v_{\alpha_i}\}\right)$.

5 MADM Problem Based on Proposed Information

In this section, we arrange the procedure of MADM techniques based on the invented techniques for evaluated operators to enhance the worth and stability of the proposed operators. For this, we consider a collection of finite alternatives $X=\{x_1,x_2,\dots,x_n\}$ and attributes $G=\{G_1,G_2,\dots,G_m\}$, where $\Theta=(\Theta_1,\Theta_2,\dots,\Theta_n)$ is the weight vector where $\Theta_i\geq 0, i=1,2,\dots,m,$ and $\sum_{i=1}^{m}\Theta_i=1$. Consider a circular intuitionistic fuzzy decision matrix $A_k=(a_{ij})_{m\times n}$ and an attribute value provided by the decision-maker e_k is $a_{ij}=\left(\mu_{ij},v_{ij};r_{ij}\right)$, which is C-IFV, where μ_{ij} is the membership degree, v_{ij} is the non-membership degree, r_{ij} is the radius and π_{ij} is the uncertainty degree. Furthermore, we aim to evaluate the normalization of the data by using the below theory, if we have cost type of data, but in the case of benefit type of data we do not aim to normalize the data, such as (64):

$$a_{ij}=\left(\mu_{ij},v_{ij};r_{ij}\right)=\begin{cases}a_{ij}, for\ benefit\ attribute\ G_i\\ a_{ij}^{c},\quad for\ cost\ attribute\ G_i\end{cases}, j=1,2,\dots,n \tag{64}$$

Where a_{ij}^{c} is the complement of a_{ij}, such that $a_{ij}^{c}=\left(v_{ij},\mu_{ij};r_{ij}\right)$, clearly, $\pi_{ij}=1-\mu_{ij}-v_{ij}$.
Finally, we evaluate or address the above procedure, we have the following steps, such as:

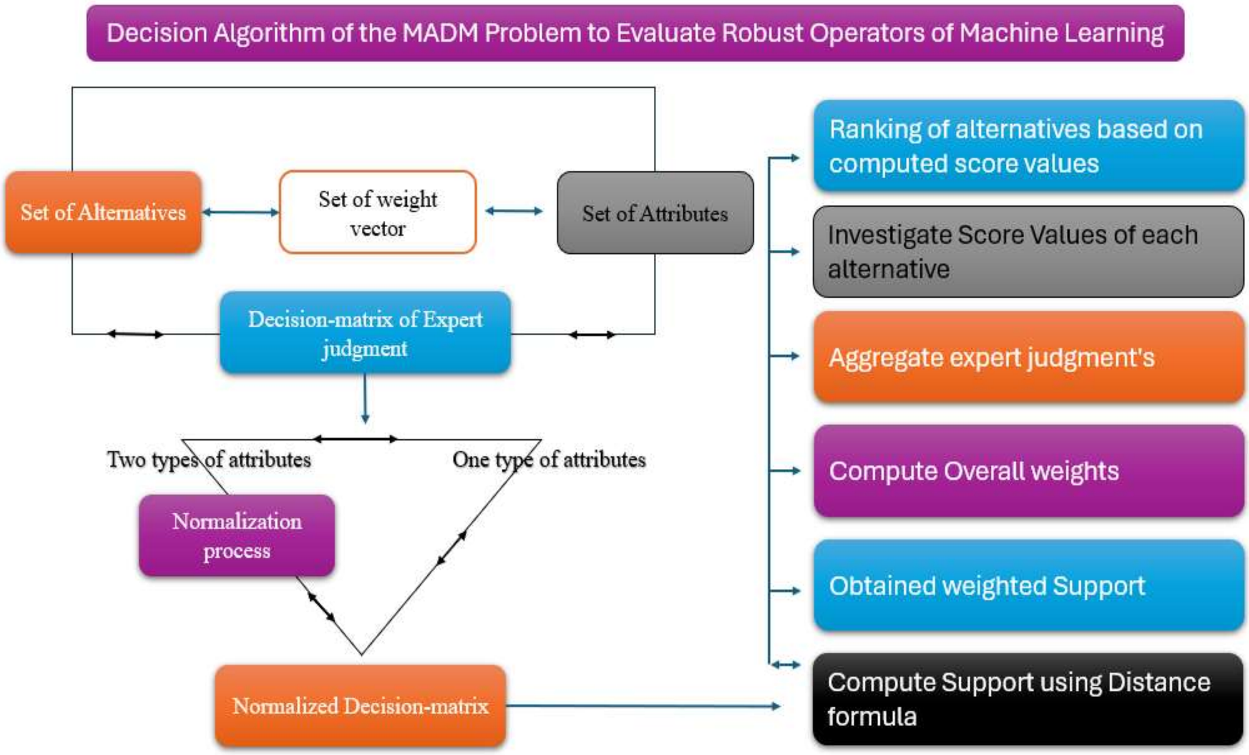

Figure 2 Diagram for the MADM Problem.

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Step 1. Determine the supports based on distance measures, such as (65-66):

$$Sup\left(a_{ij}, a_{ik}\right) = 1 - d\left(a_{ij}, a_{ik}\right), i = 1,2, \dots, n, \& j, k = 1,2, \dots m \quad (65)$$

$$d\left(a_{ij}, a_{ik}\right) = \frac{1}{2}\left(\left|\mu_{ij} - \mu_{ik}\right| + \left|v_{ij} - v_{ik}\right| + \left|r_{ij} - r_{ik}\right|\right), i = 1,2, \dots, n, \& j, k = 1,2, \dots m \quad (66)$$

Step 2. Compute weighted support as follows (67):

$$\mathcal{A}(\alpha_i) = \sum_{\substack{i=1 \\ i \neq k}}^{s} \Theta_i Sup\left(a_{ij}, a_{ik}\right) \quad (67)$$

Step 3. Calculate the overall degree of weights $\xi_{ij}, (j = 1,2, \dots, m)$ as follows (68):

$$\xi_{ij} = \frac{\Theta_j\left(1+\mathcal{A}(\alpha_j)\right)}{\sum_{j=1}^{n} \Theta_j\left(1+\mathcal{A}(\alpha_j)\right)}, i = 1,2, \dots, n, \& j, k = 1,2, \dots m \quad (68)$$

Where $\xi_{ij} \geq 0,\ j = 1,2, \dots, s, and\ \sum_{j=1}^{s} \xi_{ij} = 1$

Step 4. To aggregate human opinions, we apply derived approaches of the C-IFPWA and C-IFPWG operators as follows (69-72):

$$X_i = C - IFPWA(a_{i1}, a_{i2}, \dots, a_s)_t = \begin{pmatrix} 1 - \prod_{k=1}^{s}\left(1 - \mu_{ij}\right)^{\xi_{ij}}, \\ \prod_{j=1}^{s}\left(v_{ij}\right)^{\xi_{ij}}, \\ 1 - \prod_{k=1}^{s}\left(1 - r_{ij}\right)^{\xi_{ij}} \end{pmatrix} \quad (69)$$

$$X_i = C - IFPWA(a_{i1}, a_{i2}, \dots, a_s)_{tc} = \begin{pmatrix} 1 - \prod_{k=1}^{s}\left(1-\mu_{ij}\right)^{\xi_{ij}}, \\ \prod_{j=1}^{s}\left(v_{ij}\right)^{\xi_{ij}}, \\ \prod_{j=1}^{s}\left(r_{ij}\right)^{\xi_{ij}} \end{pmatrix} \tag{70}$$

$$X_i = C - IFPWG(a_{i1}, a_{i2}, \dots, a_s)_{t} = \begin{pmatrix} \prod_{j=1}^{s}\left(\mu_{ij}\right)^{\xi_{ij}}, \\ 1 - \prod_{j=1}^{s}\left(1-v_{ij}\right)^{\xi_{ij}}, \\ \prod_{j=1}^{s}\left(r_{ij}\right)^{\xi_{ij}} \end{pmatrix} \tag{71}$$

$$X_i = C - IFPWG(a_{i1}, a_{i2}, \dots, a_s)_{tc} = \begin{pmatrix} \prod_{j=1}^{s}\left(\mu_{ij}\right)^{\xi_{ij}}, \\ 1 - \prod_{j=1}^{s}\left(1-v_{ij}\right)^{\xi_{ij}}, \\ 1 - \prod_{j=1}^{s}\left(1-r_{ij}\right)^{\xi_{ij}} \end{pmatrix} \tag{72}$$

Step 5. Rank $X_i, (i = 1,2, \dots, n)$ in descending order by using the ranking method described in Definition 2.
Step 6. Rank all the alternatives $X_i, (i = 1,2, \dots, n)$ and select the best one following the ranking of X_i $(i = 1,2, \dots, n)$.

Furthermore, we also elaborate stepwise decision algorithm of the MADM problem in Figure 2. Figure 2 facilitates a comprehensive understanding of the aggregation process of expert's opinions.

5.1 Analysis of Machine Learning Through Proposed Operators

Machine learning (ML) analysis refers to the application of algorithms and statistical models to identify patterns, make predictions, and improve decision-making based on data. Unlike traditional programming, where explicit instructions are given to perform tasks, machine learning models learn autonomously by analysing data. ML analysis has become central to a variety of industries due to its ability to process vast amounts of data quickly and accurately. The primary objective of ML analysis is to allow systems to improve over time, enhancing their predictive accuracy and relevance in real-world applications.

Machine learning is commonly categorized into three types: supervised, unsupervised, and reinforcement learning, each with unique analytical approaches. Supervised learning is used when labelled data is available, allowing the model to learn associations between input-output pairs. This type of analysis is widely used for tasks like classification and regression, such as predicting stock prices or diagnosing diseases. Unsupervised learning is applied to unlabelled data to uncover hidden patterns or groupings. Clustering and dimensionality reduction are examples, commonly used in recommendation engines and customer segmentation. Reinforcement learning is distinct in that it involves agents learning through interactions with their environment, receiving rewards or penalties. It is often applied in game theory, robotics, and autonomous systems.

Several algorithms are fundamental to machine learning analysis, each suited to different types of data and problem domains. Linear regression and logistic regression are widely used in supervised learning for predicting numerical outcomes and binary classifications, respectively. Decision trees and random forests are used for both classification and regression tasks due to their interpretability and flexibility. Support vector machines (SVMs) are effective in high-dimensional spaces, making them useful for text classification and image recognition. In the unsupervised domain, k-means clustering and principal component analysis (PCA) are common methods for clustering data and reducing dimensionality, respectively. Neural networks and deep learning models have gained popularity for handling complex, unstructured data, such as images and audio, through layers of processing that mimic the human brain's structure.

A crucial aspect of machine learning analysis is evaluating the model's performance to ensure accuracy and reliability. For classification tasks, metrics such as accuracy, precision, recall, and F1 score are commonly used to assess the model’s effectiveness. In regression tasks, mean squared error (MSE), mean absolute error (MAE), and R-squared are popular metrics. Cross-validation techniques, such as k-fold cross-validation, help in assessing the generalizability of the model by partitioning the data into training and testing subsets. Selecting the right evaluation metric is essential as it influences how well the model meets its intended purpose and helps in comparing different models' performance accurately.

Machine learning analysis has transformative applications across diverse industries. In healthcare, ML analysis is used for predictive diagnostics, drug discovery, and personalized medicine. In finance, algorithms analyse vast datasets to detect fraud, assess credit risk, and manage investments. Retailers use machine learning to enhance customer experience through personalized recommendations and demand forecasting. The implications of these applications are profound, as ML analysis can optimize resources, reduce operational costs, and provide insights that were previously inaccessible. However, the ethical and social implications—such as data privacy, algorithmic bias, and transparency—are growing concerns that need to be addressed to ensure responsible AI usage.

Despite its rapid advancements, machine learning analysis faces several challenges. Data quality and availability can significantly impact model performance, as models trained on biased or insufficient data may yield inaccurate predictions. The complexity of some ML models, especially deep learning networks, creates a lack of interpretability, making it difficult to understand how decisions are made. This is a critical issue in fields like healthcare and finance, where accountability is essential. Looking forward, research into explainable AI (XAI) aims to make ML models more transparent and interpretable. Additionally, federated learning and privacy-preserving techniques are emerging to address privacy concerns by allowing decentralized model training on sensitive data. The future of machine learning analysis lies in building more ethical, interpretable, and adaptable models to tackle increasingly complex problems.

In this section, we evaluate the problem of machine learning through invented operators. For this, we consider four kinds of machine learning, such as $\mathbb{A}_1$: Supervised Learning, $\mathbb{A}_2$: Unsupervised Learning, $\mathbb{A}_3$: Semi-supervised Learning, $\mathbb{A}_4$: Self- supervised Learning. To select the best one, we have the following criteria, such as network impact a_1, growth analysis a_2, stock exchange impact a_3, environmental impact a_4, and the ratio of expert people in computers 5. Therefore, we utilize the C-IFPWA (or C-IFPWG) operator to develop an approach to multiple attribute group decision-making with circular intuitionistic fuzzy information, see Table 1, which involves the following steps:

Table 1. C-IF decision matrix.

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\boldsymbol{a_1}$	(0.5, 0.5; 0.9)	(0.8, 0.2; 0.1)	(0.8, 0.2; 0.6)	(0.5, 0.5; 0.2)
$\boldsymbol{a_2}$	(0.3, 0.7; 0.8)	(0.6, 0.4; 0.5)	(0.7, 0.3; 0.4)	(0.6, 0.4; 0.5)
$\boldsymbol{a_3}$	(0.4, 0.6; 0.7)	(0.7, 0.3; 0.3)	(0.4, 0.6; 0.2)	(0.9, 0.1; 0.4)
$\boldsymbol{a_4}$	(0.2, 0.8; 0.5)	(0.1, 0.9; 0.2)	(0.5, 0.5; 0.3)	(0.8, 0.2; 0.1)
$\boldsymbol{a_5}$	(0.9, 0.1; 0.1)	(0.5, 0.5; 0.3)	(0.9, 0.1; 0.9)	(0.7, 0.3; 0.8)

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Step 1. Calculate the supports, see Table 2.

Table 2. C-IF distance matrix.

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$d(\boldsymbol{a_1}, \boldsymbol{a_2})$	0.25	0.4	0.2	0.25
$d(\boldsymbol{a_1}, \boldsymbol{a_3})$	0.2	0.2	0.6	0.5
$d(\boldsymbol{a_1}, \boldsymbol{a_4})$	0.5	0.75	0.45	0.35
$d(\boldsymbol{a_1}, \boldsymbol{a_5})$	0.8	0.4	0.25	0.5
$d(\boldsymbol{a_2}, \boldsymbol{a_3})$	0.15	0.2	0.4	0.35
$d(\boldsymbol{a_2}, \boldsymbol{a_4})$	0.25	0.65	0.25	0.4
$d(\boldsymbol{a_2}, \boldsymbol{a_5})$	0.95	0.2	0.45	0.25
$d(\boldsymbol{a_3}, \boldsymbol{a_4})$	0.3	0.65	0.15	0.25
$d(\boldsymbol{a_3}, \boldsymbol{a_5})$	0.8	0.2	0.85	0.4
$d(\boldsymbol{a_4}, \boldsymbol{a_5})$	0.9	0.45	0.7	0.45

Furthermore, with the help of data in Table 2, the support matrix is listed in Table 3.

Table 3. C-IF support matrix.

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\boldsymbol{Sup(a_1, a_2)}$	0.75	0.6	0.8	0.75
$\boldsymbol{Sup(a_1, a_3)}$	0.8	0.8	0.4	0.5
$\boldsymbol{Sup(a_1, a_4)}$	0.5	0.25	0.55	0.65
$\boldsymbol{Sup(a_1, a_5)}$	0.2	0.6	0.75	0.5
$\boldsymbol{Sup(a_2, a_3)}$	0.85	0.8	0.6	0.65
$\boldsymbol{Sup(a_2, a_4)}$	0.75	0.35	0.75	0.6
$\boldsymbol{Sup(a_2, a_5)}$	0.05	0.8	0.55	0.75
$\boldsymbol{Sup(a_3, a_4)}$	0.7	0.35	0.85	0.75
$\boldsymbol{Sup(a_3, a_5)}$	0.2	0.8	0.15	0.6
$\boldsymbol{Sup(a_4, a_5)}$	0.1	0.55	0.3	0.55

Step 2. Utilize the weight $\xi_{ij}, (j = 1,2, \dots, s)$ of the decision-maker to calculate the weighted support a_{ij} of the C-IFV, and calculate the weights $\xi_{ij}, (j = 1,2, \dots, m)$ associated with the C-IFVs $a_{ij}, (j = 1,2, \dots, s)$, where $\xi_{ij} \geq 0, j = 1,2, \dots, s, and\ \sum_{j=1}^{s} \xi_{ij} = 1$, see Table 4 and Table 5.

Table 4. C-IF weighted matrix.

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\mathcal{A}(\boldsymbol{a_1})$	2.25	2.25	2.5	2.4
$\mathcal{A}(\boldsymbol{a_2})$	2.4	2.55	2.7	2.75
$\mathcal{A}(\boldsymbol{a_3})$	2.55	2.75	2	2.5
$\mathcal{A}(\boldsymbol{a_4})$	2.05	1.5	2.45	2.55
$\mathcal{A}(\boldsymbol{a_5})$	0.55	2.75	1.75	2.4

Table 5. C-IF weighted matrix with support grades.

$\boldsymbol{\xi_{ij}}$	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\mathrm{T}(\boldsymbol{\xi_1})$	0.219595	0.193452	0.213415	0.193182
$\mathrm{T}(\boldsymbol{\xi_2})$	0.22973	0.21131	0.22561	0.213068
$\mathrm{T}(\boldsymbol{\xi_3})$	0.239865	0.223214	0.182927	0.198864
$\mathrm{T}(\boldsymbol{\xi_4})$	0.206081	0.14881	0.210366	0.201705
$\mathrm{T}(\boldsymbol{\xi_5})$	0.10473	0.223214	0.167683	0.193182

Step 3. Utilize the C-IFPWA operator and C-IFPWG operator for t-norm and t-conorm, see Table 6.

Table 6. C-IF aggregated matrix.

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
C-IFPW$\boldsymbol{A_t}$	(0.474715, 0.525285, 0.732341)	(0.610975, 0.389025, 0.301826)	(0.710752, 0.289248, 0.556365)	(0.739266, 0.260734, 0.464495)

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
C-IFPW$\boldsymbol{A}_{tc}$	(0.474715, 0.525285, 0.580452)	(0.610975, 0.389025, 0.254388)	(0.710752, 0.289248, 0.630966)	(0.739266, 0.260734, 0.317153)
C-IFPW$\boldsymbol{G}_t$	(0.371096, 0.628904, 0.580452)	(0.482829, 0.517171, 0.254388)	(0.631797, 0.368203, 0.630966)	(0.685499, 0.314501, 0.317153)
C-IFPW$\boldsymbol{G}_{tc}$	(0.371096, 0.628904, 0.732341)	(0.482829, 0.517171, 0.301826)	(0.631797, 0.368203, 0.556365)	(0.685499, 0.314501, 0.464495)

Step 5. Rank $\mathbb{A}_j$ $(j = 1,2, \dots, n)$ in descending order by using the ranking method described in Definition 2, see Table 7.

Table 7. C-IF score values information.

	s_{α_t}	$s_{\alpha_{tc}}$	s_{α_t}	$s_{\alpha_{tc}}$
$\mathbb{A}_1$	-0.0370	0.0670	0.2345	0.2223
$\mathbb{A}_2$	-0.0294	0.0565	0.2660	0.1518
$\mathbb{A}_3$	-0.1497	-0.0087	0.1663	0.1177
$\mathbb{A}_4$	-0.1888	-0.0104	0.1467	0.1723

Step 6. Rank all the alternatives $\mathbb{A}_j (j = 1,2, \dots, n)$ and select the best one following the ranking of $\mathbb{A}_j$ $(j = 1,2, \dots, n)$, see Table 8.

Table 8. C-IF ranking matrix.

Methods	Ranking values
C-IFPW$\boldsymbol{A}_t$	$\mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_3 > \mathbb{A}_4$
C-IFPW$\boldsymbol{A}_{tc}$	$\mathbb{A}_1 > \mathbb{A}_2 > \mathbb{A}_3 > \mathbb{A}_4$
C-IFPW$\boldsymbol{G}_t$	$\mathbb{A}_2 > \mathbb{A}_1 > \mathbb{A}_3 > \mathbb{A}_4$
C-IFPW$\boldsymbol{G}_{tc}$	$\mathbb{A}_1 > \mathbb{A}_4 > \mathbb{A}_2 > \mathbb{A}_3$

The most valuable decision is $\mathbb{A}_2$ according to the theory of C-IFPWA and C-IFPWG operators with t-norm, but according to the theory of C-IFPWA and C-IFPWG operators with t-conorm, we obtain the best decision is $\mathbb{A}_1$.

6 Comparative Study

In this section, we compare the derived techniques or methods with various old or existing methods based on the data in Table 1. For this, we needed to select some prevailing operators based on IFSs and circular IFSs, then in the presence of the information in Table 1, we discussed their ranking results in Table 9. For this, we have the following existing techniques, such as simple average operator (AVO) for IFST was exposed by Xu [34]. Moreover, the geometric operator (GEO) for IFST was exposed by Xu and Yager [62]. In 2018, Jiang et al. [36] derived the power AVO and power GEO for IFST. Garg [63] developed Einstein AVO and GEO for the decision analysis process. Jana and Pal [64] proposed picture fuzzy AVO and GEO using operations of Dombi t-norms and t-conorms. Al-Quran [65] modified the theory of t-spherical hesitant fuzzy information and decision analysis process. Hence the comparative analysis is listed in Table 9 in the consideration of the data in Table 1.

The most valuable decision is $\mathbb{A}_2$ according to the theory of C-IFPWA and C-IFPWG operators with t-norm, but according to the theory of C-IFPWA and C-IFPWG operators with t-conorm, we obtain the best decision is $\mathbb{A}_1$. Anyhow, some existing techniques are working accurately because of limitations and vagueness due to their structure. Hence, the proposed operators based on C-IFS are novel and no one can derive them. Therefore, the derived operators are superior to existing techniques.

Table 9. Comparative analysis of the proposed and existing techniques.

Methods	Score values	Ranking values
Xu [34]	$---bounded---$	$---bounded---$
Xu and Yager [62]	$---bounded---$	$---bounded---$
Jiang et al. [36]	$---bounded---$	$---bounded---$
Garg [63]	$---bounded---$	$---bounded---$
Jana and Pal [64]	$---bounded---$	$---bounded---$
Al-Quran [65]	$---bounded---$	$---bounded---$
C-IFPW$\boldsymbol{A_t}$	$-0.037, -0.09, -0.14, -0.18$	$A_2 > A_1 > A_3 > A_4$
C-IFPW$\boldsymbol{A_{tc}}$	$0.066, 0.056, -0.008, -0.010$	$A_1 > A_2 > A_3 > A_4$
C-IFPW$\boldsymbol{G_t}$	$0.23, 0.26, 0.16, 0.14$	$A_2 > A_1 > A_3 > A_4$
C-IFPW$\boldsymbol{G_{tc}}$	$0.22, 0.15, 0.11, 0.17$	$A_1 > A_4 > A_2 > A_3$

7 Conclusion

The decision analysis process is used to investigate suitable optimal options under various characteristics or attribute information about human opinions. Although, various research scholars developed many mathematical approaches and decision-making methods using different fuzzy domains of FSs and IFSs. Sometimes decisions are unable to address incomplete and uncertain information due to the restricted and limited structure of discussed mathematical models. The aims of this article are characterised as follows:

1) Some flexible operations are formulated under the system of C-IFSs.
2) We derived power aggregation operators of the C-IFPWA, C-IFPWOA, C-IFPWG, and C-IFPWOG operators with some prominent properties.
3) We evaluated the procedure of the MADM technique for resolving the utilization of the most suitable part of machine learning in complicated scenarios.
4) We illustrated some numerical examples for addressing the comparison between proposed techniques and existing methods to show the effectiveness and reliability of the presented operators.
5) A comparative study presented a contrasting technique for comparing the results of existing approaches with pioneered mathematical approaches.

In the coming future, we aim to evaluate the power operators for circular pythagorean fuzzy sets, circular q-rung orthopair fuzzy sets, and their extensions. Moreover, we can also apply the derived approaches of this article to get flexible solutions from complicated real-life applications and numerical examples. Additionally, we will prove their supremacy with the help of some applications discussed in artificial intelligence, neural networks, machine learning, and game theory.

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