NON-LINEAR INTERVAL-VALUED INTUITIONISTIC FUZZY NUMBER (IVIFN) APPROACH FOR AN EPQ MODEL WITH OPTIMAL INVESTMENT IN DEFECTIVE ITEMS CONSIDERING LEARNING EFFECTS

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Abstract:

In recent years, research has primarily focused on addressing imprecision in linear forms; however, uncertainty often occurs in non-linear forms as well. This study extends the concept of Linear Interval-Valued Intuitionistic Fuzzy Numbers (LIVIFN) to Non-Linear Interval-Valued Intuitionistic Fuzzy Numbers (NLIVIFN), establishing their formulation and parametric structure along with their logical significance. Different geometric representations of NLIVIFN are analysed and classified. Furthermore, an intuitification technique is developed, which holds significant value for improving crispification skills. A realistic example is presented to demonstrate the impact of NLIVIFN on an Economic Production Quantity (EPQ) model, focusing on an imperfect product with learning and reworking of defective items. A procedure is introduced to determine the optimal shipment size and defective percentage by minimizing the average expected total cost. Results indicate that investment in learning leads to a 98% recovery rate of defective items, providing economic benefits to manufacturers. Additionally, a 50% increase in demand stimulates learning, increasing production by 36% and reducing defective item production by 51%. Finally, a comparative analysis underscores the value of this novel work, showcasing its effectiveness in addressing non-linear uncertainties and enhancing production processes, cost efficiency, and decision-making in supply chain management.

1 Introduction

In this decade, the knowledge of vagueness plays a crucial role in various research arena and researchers from distinct fields like marketing, finance, science and technology, social media, clinical etc. has incorporated the concept of impreciseness in their respective domain. In 1965, Prof. Zadeh [1] manifested a amazing perception of the fuzzy set theory. Further, Chang & Zadeh [2] ignited the formation of fuzzy construction and since then, several works have been established in this research arena [3, 4, 5] with the innumerable up gradation and improvement of postulations of fuzzy set theory, the subject in due course became a matter of enormous academic concern. As research in this arena has proceeded, the conception of vagueness is stretched out into interval-valued fuzzy sets [6]. Fuzzy set doesn't consider the degree of hesitation, that is degree of non-belongingness**.** In 1986, Prof. Atanassov [7] explored the proposal of intuitionistic fuzzy set considering both membership and non-membership functions of the fuzzy number. Further, as an extension of intuitionistic fuzzy set, Zhang et al. [8] introduced interval-valued intuitionistic fuzzy number and its application. Also, researchers developed arithmetic operation [9], assignment problem [10], similarity measure and score

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function [11] in interval-valued intuitionistic fuzzy arena. Additionally, some multiple attribute decision making problem has been solved in interval-valued intuitionistic domain using some useful operators like i) aggregation operator [12] ii) exponential operator [13]. To Sum up, mainly MCDM methods are done based on similarity measures [14,15], inclusion measure [16], entropy measure [17,18], cross-entropy measure [19] and distance measures [20]. Instead of these techniques, researchers are also applied linguistic intuitionistic fuzzy power Bonferroni Mean operators [21], fuzzy generalized aggregation operator [22], Hamacher aggregation operators [23], hybrid weighted aggregation operators [24, 25], Hamacher ordered weighted geometric operator [26], fuzzy prioritized hybrid weighted aggregation operator [27] to solve MCDM or MADM problem. Further, some special suitable techniques are incorporated in interval valued intuitionistic domain like evidential reasoning methodology [28], particle swarm optimization techniques [29], VIKOR method [30], transform technique [31] to solve lots of decision-making problem.

The basic reasons for using non-linear interval valued fuzzy number in place of linear interval valued fuzzy number are also explained, introducing the idea of non-linear and generalized interval valued intuitionistic fuzzy numbers. Previously, the researcher used to consider takes the maximum value of truth, falsity function as 1, however the fixed value concept must be abandoned. The viewpoint of different decision makers at any point on a fixed scale is a certain quantity which is less than one (but the quantity belongs to the zero to one interval). Thus, we make a generalized definition of NLIVIFN for all type of decision makers, and construct the generalized number from this viewpoint.

The classical EPQ model Silver et al. [32] assumes that the manufacturing process is non-defective, and all the items produced are of perfect quality. However, in real world, it is observed that the defective items are produced due to various reasons such as machinery breakdown, human error, default in production processes, etc. Thus, these defective items are either rejected or repaired and reworked, and these leads to essential changes in corresponding total costing of the inventory system. Recently, numerous researchers are working on EPQ/EOQ models with imperfect quality items. Khouja and Mehrez [33] formulated and solved an economic production lot size problem of an imperfect production process. This was further triggered by Salameh & Jaber [34]. They developed an EOQ model to determine the optimal lot size which contains imperfect items and those are sold in a secondary market at a discounted price. El-Kassar [35] examined an EOQ model with imperfect quality items, where the imperfect quality items are sold at a discounted price. Recently, the researcher are does not scrap the defective items but rework it for further use. Buscher and Lindner [36] have presented a procedure which synchronized the determination of production as well as rework in batch sizes. Jawlaa and Singh [37] developed a model where the production of imperfect items are reworked under preservation technology and learning. Glock and Jaber [38] worked on multi echelon production model with learning, forgetting, and defective items are rework, and twice defective items are scraped. Investment in inventory is essential to upgrade the machinery which will reduce the production of defective items. Dey and Giri [39] investigated a model with investment to reduce the production of defective items. Giri and Glock [40] has addressed on closed loop supply chain model with learning and forgetting during manufacturing and re manufacturing of the returned items. Deng-Maw TSAI & Ji-Cheng WU [41] focused on economic production quantity learning and rework; Wright, T.P [42] manifested factor effecting cost detection model; Evan L. Porteus [43] introduced optimal lot sizing model ; Sahoo et al. [44] focused on MCDM model based on trends and insights; Ali & Hussain [45] developed MADM model on intuitionistic fuzzy soft information; Sing & Sarkar [46] focused on fuzzy linear equation model; Biswas et al. [47] introduced a new MCGDM model on shopping field; Wang et al. [48] manifested complex intuitionistic fuzzy based DOMBI operator and its application on green supplier selection. Apart from this work several researchers focused on [49-53] decision making problem under uncertain environment.

In this paper, we developed the theoretical and graphical knowledge of NLIVIFN along with the conception of parametric(α, β) representations, arithmetic operations and a new Intuitification technique of NLIVIFN. Further, the paper investigates EPQ model with constant demand and imperfect production process where the imperfect items are reworked. Here, the workers experience due to learning is considered in NLIVIFN environment. The manufacturer invests money to improve the production quality and reduce the number of defective items. Also in practice, the time required to produce depend on the number of items produced by the worker. Thus, the production time and hence the production cost can’t be treated as a constant. Thus, the paper focused on the effect of investment for an imperfect product with learning and reworking of defective items on an EPQ model. A procedure is developed to determine the optimal decisions i.e., optimizing the total cost.

1.1 Motivation

In general, if we consider a fuzzy number then the degree of acceptance membership function is being considered only. Now obviously when we deal with uncertainty then there should be a concept of non-belongingness (i.e., the concept of intuitionistic fuzzy number). Now if the components are lying in an interval, then how it is looks like? What will be the graphical figure? How can we define non-linear intuitionistic fuzzy number? How can we establish the concept of ranking and de-intuitification of a NLIVIFN? How will we apply it in real life problem? Aiming at these points we started to build up this article.

1.2 Novelties

Although there exist several articles where interval valued intuitionistic fuzzy sets and number are defined and apply to various field. But still there are some crucial works is developed here in this article.

(i) Formation of NLIVIFN in easier manner.
(ii) Graphical classification of triangular NLIVIFN.
(iii) De-intuitification of a triangular NLIVIFN.
(iv) Suitable application in real life inventory control problem.

1.3 Practical and Methodological Aims of the Study

1.3.1 Practical Aims of the Study:

- **Optimize Investment in Defective Items:** The study aims to provide a practical framework for determining the optimal investment in handling defective items within an Economic Production Quantity (EPQ) model, which is commonly used in production and inventory management.
- **Improve Decision-Making with Learning Effects:** The goal of the research is to enhance decision-making procedures in production systems by integrating learning effects. It attempts to demonstrate how learning might eventually lower the cost of defective items, improving operational efficiency.
- **Application of Fuzzy Logic in Uncertain Environments:** The study intends to apply interval-valued intuitionistic fuzzy numbers (IVIFN) to address uncertainties in production, particularly regarding defective items. This approach can help practitioners manage ambiguity and vagueness in real-world situations.

1.3.2 Methodological Aims of the Study:

- **Develop a Non-Linear EPQ Model:** The study aims to formulate a non-linear EPQ model that incorporates interval-valued intuitionistic fuzzy numbers (IVIFN) to represent uncertainty and imprecision in production parameters, such as defect rates.
- **Apply Intuitification Techniques:** The methodology aims to improve the precision of the fuzzy numbers by applying "intuitification" techniques, making the model more resilient and precise for handling real-world data.
- **Incorporate Learning Effects into EPQ Model:** The study aims to incorporate learning curve effects into the EPQ model to account for improvements in production processes over time, which reduce defect rates and costs associated with defective items.
- **Optimization of Production and Investment Decisions:** The goal of the study is to create an optimisation method that strikes a balance between investment, defect control, and production quantity to guarantee optimal performance in uncertain and evolving environments.

1.4 Organization of the Paper

The article has been divided into seven sections. In section 2, we presented some mathematical preliminaries. In section 3, we presented the different form and characters of Interval valued fuzzy sets and number and algebraic properties. In Section 4, we focused on Intuitification of NLIVIFN. In Section 5, the innovative model development and analysis have been addressed. In Section 6, numerical study along with sensitivity analysis and comparative study have been discussed. Finally, the conclusion has been shown in Section 7.

2 Mathematical Preliminaries

Definition 2.1: Fuzzy Set: [1] A set $\widetilde{M}$, defined as $\widetilde{M} = \{(\alpha, \mu_{\widetilde{M}}(\alpha)): \alpha \in M, \mu_{\widetilde{M}}(\alpha) \in [0,1]\}$, where $\mu_{\widetilde{M}}(\alpha)$ denotes the membership function of $\widetilde{M}$, is called a fuzzy set.

Definition 2.2: *Intuitionistic Fuzzy Set:* [7] Let a set X be fixed. An IFS $\tilde{A}^i$ in X is an object having the form $\tilde{A}^i = \{< x, \mu_{\tilde{A}^i}(x), \vartheta_{\tilde{A}^i}(x) >: x \in X\}$, where the $\mu_{\tilde{A}^i}(x): X \to [0,1]$ and $\vartheta_{\tilde{A}^i}(x): X \to [0,1]$ define the degree of membership and degree of non-membership respectively, of the element $x \in X$ to the set $\tilde{A}^i$, which is a subset of X, for every element of $x \in X$ (1):

$$0 \leq \mu_{A^{\sim i}}(x) + \vartheta_{A^{\sim i}}(x) \leq 1 \tag{1}$$

Definition 2.3: Intuitionistic fuzzy number: [7] An intuionistic fuzzy number$\check{A}$ is defined as follows:
1) an intuitionistic fuzzy subset of the real line
2) normal, i.e for x∈R such that $\mu_{\check{A}}(x)=1$, hence $v_{\check{A}}(x) = 0$
3) a convex set for the membership function $\mu_{\check{A}}(x)$
For $x_1, x_2 \in R$, $\mu_{\check{A}}(\lambda x_1+(1-\lambda) x_2) \geq \min(\mu_{\check{A}}(x_1),\mu_{\check{A}}(x_2))$ where $0<\lambda<1$
4) a concave set for the membership function $v_{\check{A}}(x)$
For $x_1, x_2 \in R$, $v_{\check{A}}(\lambda x_1+(1-\lambda) x_2) \leq \max(v_{\check{A}}(x_1), v_{\check{A}}(x_2))$ where $0<\lambda<1$

3 Interval valued fuzzy sets and number

Definition 3.1: Interval-Valued Intuitionistic Fuzzy set: [8] An interval valued fuzzy set $\tilde{A}$ on R is defined by (2):

$$\tilde{A} = \left[\left\{x, \left(\mu_{\tilde{A}^U}(x), \mu_{\tilde{A}^L}(x)\right), \left(\vartheta_{\tilde{A}^U}(x), \vartheta_{\tilde{A}^L}(x)\right)\right\}: x \in R\right] \tag{2}$$

Where $x \in R$ and $\mu_{\tilde{A}^U}(x), \mu_{\tilde{A}^L}(x)\vartheta_{\tilde{A}^U}(x), \vartheta_{\tilde{A}^L}(x)$, maps R into [0,1] and $\forall x \in R$, $\mu_{\tilde{A}^L}(x) \leq \mu_{\tilde{A}^U}(x)$ and $\vartheta_{\tilde{A}^U}(x) \geq \vartheta_{\tilde{A}^L}(x)$.

Definition 3.2: Linear Interval-Valued triangular intuitionistic fuzzy number: An interval-valued triangular fuzzy number is denoted by

$$\tilde{A}_{linvi} = [\{(a_1, b, c_1; \lambda), (a, b, c; \omega)\}, \{(d_1, b, e_1; \delta), (d, b, e; \mu)\}] \tag{3}$$

Where $0 < \omega \leq \lambda \leq 1, 0 < \delta \leq \mu \leq 1$and $a_1 < a < b < c < c_1$also $0 < \omega + \delta \leq 1$ and $0 < \lambda + \mu \leq 1$.
Where the upper and lower membership function is defined by

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \lambda\left(\frac{x-a_1}{b-a_1}\right)^{\cdot}, & a_1 \leq x \leq b \\ \lambda\left(\frac{c_1-x}{c_1-b}\right)^{\cdot}, & b \leq x \leq c_1 \\ 0, & otherwise \end{cases} \tag{4}$$

and

$$\mu_{\tilde{A}^L}(x) = \begin{cases} \omega\left(\frac{x-a}{b-a}\right)^{\cdot}, & a \leq x \leq b \\ \omega\left(\frac{c-x}{c-b}\right)^{\cdot}, & b \leq x \leq c \\ 0, & otherwise \end{cases} \tag{5}$$

And non membership functions are defined as

$$\vartheta_{\tilde{A}^U}(x) = \begin{cases} \delta\left(\frac{b-x}{b-d_1}\right)^{\cdot}, & d_1 \le x \le b \\ \delta\left(\frac{x-b}{e_1-b}\right)^{\cdot}, & b \le x \le e_1 \\ 1, & otherwise \end{cases} \tag{6}$$

and

$$\vartheta_{\tilde{A}^L}(x) = \begin{cases} \mu\left(\frac{b-x}{b-d}\right)^{\cdot}, & d \le x \le b \\ \mu\left(\frac{x-b}{e-b}\right)^{\cdot}, & b \le x \le e \\ 1, & otherwise \end{cases} \tag{7}$$

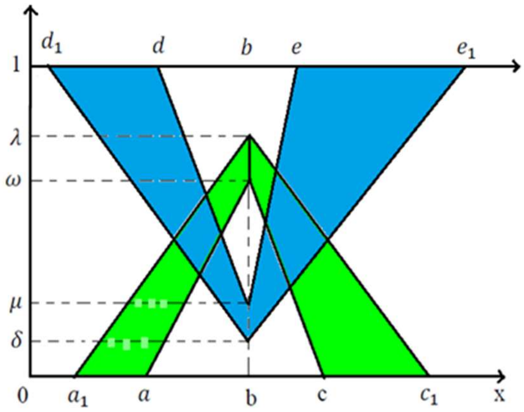

Figure 1.: Interval valued triangular intuitionistic fuzzy number [8]

Definition 3.3: (α, β)-cut of Interval-Valued triangular intuitionistic fuzzy number: α-cut of interval-valued triangular fuzzy number

$$\tilde{A}_{linvi} = [\{(a_1, b, c_1; \lambda), (a, b, c; \omega)\}, \{(d_1, b, e_1; \delta), (d, b, e; \mu)\}] \tag{8}$$

is denoted by

$$\left(\tilde{A}_{linvi}\right)_{(\alpha,\beta)} = \left[\{A_l^U(\alpha_2), A_r^U(\alpha_2); A_l^L(\alpha_1), A_r^L(\alpha_1)\}, \{A_l^U(\beta_2), A_r^U(\beta_2); A_l^L(\beta_1), A_r^L(\beta_1)\}\right] \tag{9}$$

Where,
$$A_l^U(\alpha_2) = a_1 + \frac{\alpha_1}{\lambda}(b - a_1), A_r^U(\alpha_2) = c_1 - \frac{\alpha_1}{\lambda}(c_1 - b)$$
$$A_l^L(\alpha_1) = a + \frac{\alpha_2}{\omega}(b - a), A_r^L(\alpha_1) = c - \frac{\alpha_2}{\omega}(c - b)$$
$$A_l^U(\beta_2) = b - \frac{\beta_2}{\delta}(b - d_1), A_r^U(\beta_2) = b + \frac{\beta_2}{\delta}(e_1 - b)$$
$$A_l^L(\beta_1) = b + \frac{\beta_1}{\mu}(b - d), A_r^L(\beta_1) = b - \frac{\beta_1}{\mu}(e - b)$$

also $0 \le \alpha_1 \le 1, 0 \le \alpha_2 \le 1$, also $0 \le \beta_1 \le 1, 0 \le \beta_2 \le 1$and $\alpha_1 \le \alpha_2, \beta_1 \le \beta_2, 0 \le \alpha_1 + \beta_1 \le 1, 0 \le \alpha_2 + \beta_2 \le 1$.
Note: If $a_1 = a$, $c_1 = c$ and $\alpha_1 = \alpha_2$ then it is a triangular fuzzy number.

Definition 3.4: Non-linear Interval-Valued triangular intuitionistic fuzzy number: An interval-valued triangular fuzzy number is denoted by

$$\tilde{A}_{ninvi} = [\{(a_1, b, c_1; \lambda; p_1, p_2), (a, b, c; \omega; q_1, q_2)\}, \{(d_1, b, e_1; \delta; r_1, r_2), (d, b, e; \mu; s_1, s_2)\}] \tag{10}$$

Where $0 < \omega \le \lambda \le 1$, $0 < \delta \le \mu \le 1$ and $a_1 < a < b < c < c_1$ also $0 < \omega + \delta \le 1$ and $0 < \lambda + \mu \le 1$.

Where the upper and lower membership function is defined by

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \lambda\left(\frac{x-a_1}{b-a_1}\right)^{p_1}, & a_1 \le x \le b \\ \lambda\left(\frac{c_1-x}{c_1-b}\right)^{p_2}, & b \le x \le c_1 \\ 0, & otherwise \end{cases}, \quad \mu_{\tilde{A}^L}(x) = \begin{cases} \omega\left(\frac{x-a}{b-a}\right)^{q_1}, & a \le x \le b \\ \omega\left(\frac{c-x}{c-b}\right)^{q_2}, & b \le x \le c \\ 0, & otherwise \end{cases} \tag{11}$$

And non membership functions are defined as

$$\vartheta_{\tilde{A}^U}(x) = \begin{cases} \delta\left(\frac{b-x}{b-d_1}\right)^{r_1}, & d_1 \le x \le b \\ \delta\left(\frac{x-b}{e_1-b}\right)^{r_2}, & b \le x \le e_1 \\ 1, & otherwise \end{cases}, \quad \vartheta_{\tilde{A}^L}(x) = \begin{cases} \mu\left(\frac{b-x}{b-d}\right)^{s_1}, & d \le x \le b \\ \mu\left(\frac{x-b}{e-b}\right)^{s_2}, & b \le x \le e \\ 1, & otherwise \end{cases} \tag{12}$$

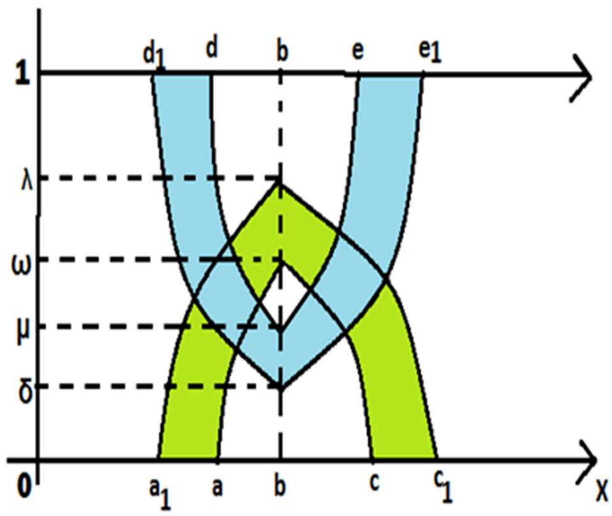

Figure 2.: NLIVIFN (Triangular) where $p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2 > 1$

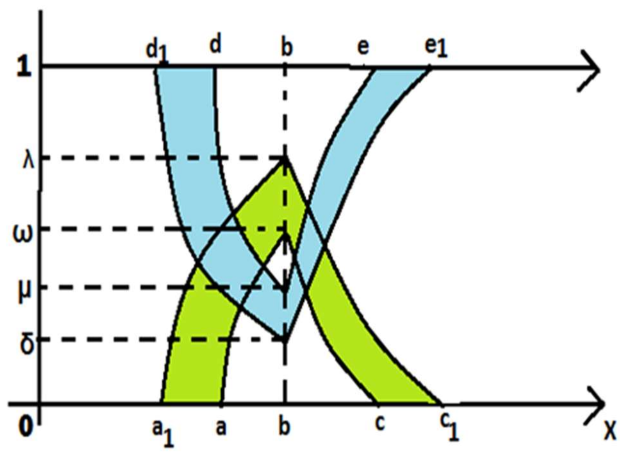

Figure 3.: NLIVIFN (Triangular) where $p_1, q_1, r_1, s_2 > 1$ and $q_2, p_2, r_2, s_2 < 1$.

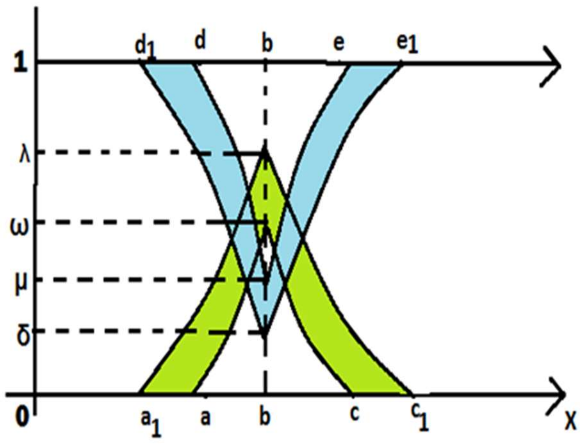

Figure 4.: NLIVIFN (Triangular) where $p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2 < 1$

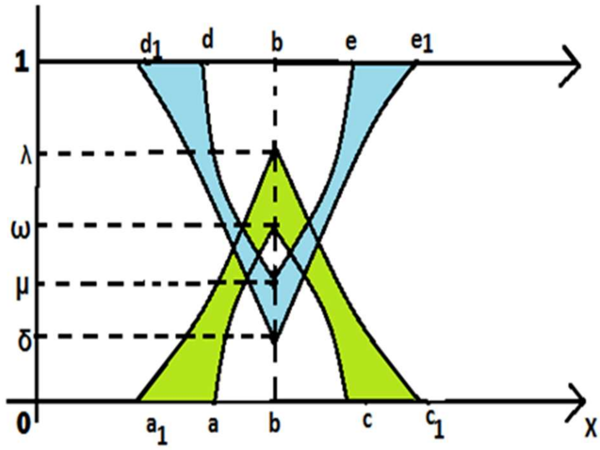

Figure 5.: NLIVIFN (Triangular) where $p_1, p_2, r_1, r_2 < 1$ *and* $q_1, q_2, s_1, s_2 > 1$

Definition 3.5: (α, β)*-cut of Non-Linear Interval-Valued triangular intuitionistic fuzzy number:*α-cut of interval-valued triangular fuzzy number (13):

$$\tilde{A}_{ninvi} = [\{(a_1, b, c_1; \lambda; p_1, p_2), (a, b, c; \omega; q_1, q_2)\}, \{(d_1, b, e_1; \delta; r_1, r_2), (d, b, e; \mu; s_1, s_2)\}] \tag{13}$$

is denoted by (14):

$$\left(\tilde{A}_{ninvi}\right)_{(\alpha,\beta)} = \left[\left\{A_l^U(\alpha_2), A_r^U(\alpha_2); A_l^L(\alpha_1), A_r^L(\alpha_1)\right\}, \left\{A_l^U(\beta_2), A_r^U(\beta_2); A_l^L(\beta_1), A_r^L(\beta_1)\right\}\right] \tag{14}$$

Where, $$A_l^U(\alpha_2) = a_1 + \left(\frac{\alpha_1}{\lambda}\right)^{\frac{1}{p_1}}(b - a_1),\ A_r^U(\alpha_2) = c_1 - \left(\frac{\alpha_1}{\lambda}\right)^{\frac{1}{p_2}}(c_1 - b)$$

$$A_l^L(\alpha_1) = a + \left(\frac{\alpha_2}{\omega}\right)^{\frac{1}{q_1}}(b - a),\ \ A_r^L(\alpha_1) = c - \left(\frac{\alpha_2}{\omega}\right)^{\frac{1}{q_2}}(c - b)$$

$$A_l^U(\beta_2) = b - \left(\frac{\beta_2}{\delta}\right)^{\frac{1}{r_1}}(b - d_1),\ \ A_r^U(\beta_2) = b + \left(\frac{\beta_2}{\delta}\right)^{\frac{1}{r_2}}(e_1 - b)$$

$$A_l^L(\beta_1) = b - \left(\frac{\beta_1}{\mu}\right)^{\frac{1}{s_1}}(b - d),\ \ A_r^L(\beta_1) = b + \left(\frac{\beta_1}{\mu}\right)^{\frac{1}{s_2}}(e - b)$$

also $0 \le \alpha_1 \le 1, 0 \le \alpha_2 \le 1$, also $0 \le \beta_1 \le 1, 0 \le \beta_2 \le 1$ and $\alpha_1 \le \alpha_2, \beta_1 \le \beta_2, 0 \le \alpha_1 + \beta_1 \le 1, 0 \le \alpha_2 + \beta_2 \le 1$.

Example: Let us consider a non-linear interval valued intuitionistic triangular fuzzy number as

$$\tilde{A}_{ninvi} = \left[\left\{(20,25,30;1;3,2),\left(22,25,28;0.8;1,\tfrac{1}{2}\right)\right\},\left\{\left(20,25,29;0;\tfrac{2}{3},2\right),\left(24,25,28;0.2;\tfrac{1}{4},\tfrac{3}{5}\right)\right\}\right] \quad (15)$$

The (α,β)-cut is denoted by,

$$\left(\tilde{A}_{ninvi}\right)_{(\alpha,\beta)} = \left[\left\{A_l^U(\alpha_2), A_r^U(\alpha_2); A_l^L(\alpha_1), A_r^L(\alpha_1)\right\},\left\{A_l^U(\beta_2), A_r^U(\beta_2); A_l^L(\beta_1), A_r^L(\beta_1)\right\}\right] \quad (16)$$

Where,

$$A_l^U(\alpha_2) = 20 + 5(\alpha_1)^{\frac{1}{3}},\ A_r^U(\alpha_2) = 30 - 5(\alpha_1)^{\frac{1}{2}}$$
$$A_l^L(\alpha_1) = 22 + \frac{15\alpha_2}{4},\ A_r^L(\alpha_1) = 28 - 3\left(\frac{5\alpha_2}{4}\right)^2$$
$$A_l^U(\beta_2) = 25 - 5(\beta_2)^{\frac{3}{2}},\ A_r^U(\beta_2) = 25 + 4(\beta_2)^{\frac{1}{2}}$$
$$A_l^L(\beta_1) = 25 - (5\beta_1)^4,\ A_r^L(\beta_1) = 25 + 3\left(\frac{5\beta_1}{3}\right)^{\frac{5}{3}}$$

Table 1: Numerical Results

$\alpha_1, \alpha_2, \beta_1, \beta_2$	$A_l^U(\alpha_2)$	$A_r^U(\alpha_2)$	$A_l^L(\alpha_1)$	$A_r^L(\alpha_1)$	$A_l^U(\beta_2)$	$A_r^U(\beta_2)$	$A_l^L(\beta_1)$	$A_r^L(\beta_1)$
0	20.0000	30.0000	22.0000	28.0000	25.0000	25.0000		
0.1	22.3208	28.4189	22.3750	27.9531	24.8419	26.2649		
0.2	22.9240	27.7639	22.7500	27.8125	24.5528	26.7889	25.0000	25.0000
0.3	23.3472	27.2614	23.1250	27.5781	24.1784	27.1909	24.9998	25.0938
0.4	23.6840	26.8377	23.5000	27.2500	23.7351	27.5298	24.9961	25.2976
0.5	23.9685	26.4645	23.8750	26.8281	23.2322	27.8284	24.9802	25.5850
0.6	24.2172	26.1270	24.2500	26.3125	22.6762	28.0984	24.9375	25.9449
0.7	24.4395	25.8167	24.6250	25.7031	22.0717	28.3466	24.8474	26.3706
0.8	24.6416	25.5279	25.0000	25.0000	21.4223	28.5777	24.6836	26.8573
0.9	24.8274	25.2566			20.7309	28.7947	24.4138	27.4014
1	25.0000	25.0000			20.0000	29.0000	24.0000	28.0000

4 Intuitification of NLIVIFN:

Intuitification method is an essential technique for a fuzzy problem from two crucial viewpoints (i) those who are not familiar with the idea of fuzzy, they can relate the solution, (ii) What is the crispified significance of the intuitionistic solution. Intuitification is the process of producing a scientific result in intuitionistic logic that relate with crisp number. There are several intuitification techniques among them some common and useful aids are as follows:
(1) Centre of Area (COA) method
(2) Bisector of Area (BOA) method
(3) Largest of Maxima (LOM) method
(4) Smallest of Maxima (SOM) Method
(5) Mean of Maxima (MOM) method:

4.1 Alpha-Beta Cut (α/β) Method:

Here we developed an intuitification method for NLIVIFN based on its parametric representation defined as follows (5):

$$\mathrm{D} = \int_{\alpha_1=0}^{\lambda} \frac{\left(L^{-1}(\alpha_1) + R^{-1}(\alpha_1)\right)d\alpha_1}{2} - \int_{\alpha_2=0}^{\omega} \frac{\left(L^{-1}(\alpha_2) + R^{-1}(\alpha_2)\right)d\alpha_2}{2} +$$

$$\int_{\beta_2=\delta}^{1} \frac{\left(L^{-1}(\beta_2)+R^{-1}(\beta_2)\right)d\beta_2}{2} \int_{\beta_1=\mu}^{1} \frac{\left(L^{-1}(\beta_1)+R^{-1}(\beta_1)\right)d\beta_1}{2} = \int_{\alpha_1=0}^{\lambda} \frac{\left(a_1+\left(\frac{\alpha_1}{\lambda}\right)^{\frac{1}{p_1}}(b-a_1)+c_1-\left(\frac{\alpha_1}{\lambda}\right)^{\frac{1}{p_2}}(c_1-b)\right)d\alpha_1}{2} -$$

$$\int_{\alpha_2=0}^{\omega} \frac{\left(a+\left(\frac{\alpha_2}{\omega}\right)^{\frac{1}{q_1}}(b-a)+c-\left(\frac{\alpha_2}{\omega}\right)^{\frac{1}{q_2}}(c-b)\right)d\alpha_2}{2} +$$

$$\int_{\beta_2=\delta}^{1} \frac{\left(b-\left(\frac{\beta_2}{\delta}\right)^{\frac{1}{r_1}}(b-d_1)+b+\left(\frac{\beta_2}{\delta}\right)^{\frac{1}{r_2}}(e_1-b)\right)d\beta_2}{2} - \int_{\beta_1=\mu}^{1} \frac{\left(b-\left(\frac{\beta_1}{\mu}\right)^{\frac{1}{s_1}}(b-d)+b+\left(\frac{\beta_1}{\mu}\right)^{\frac{1}{s_2}}(e-b)\right)d\beta_1}{2} =$$

$$\frac{1}{2}\left\{\begin{array}{c} a_1\lambda + (b-a_1)\frac{\lambda p_1}{1+p_1} + c_1\lambda - (c_1-b)\frac{\lambda p_2}{1+p_2} - \\ -a\omega - (b-a)\frac{\omega q_1}{1+q_1} - c\omega + (c-b)\frac{\omega q_2}{1+q_2} + 2b(1-\delta) \\ -\left(\frac{1}{\delta}\right)^{\frac{1}{r_1}} \frac{(b-d_1)(1-\delta)^{\frac{1}{r_1}+1}}{\frac{1}{r_1}+1} + \left(\frac{1}{\delta}\right)^{\frac{1}{r_2}} \frac{(e_1-b)(1-\delta)^{\frac{1}{r_2}+1}}{\frac{1}{r_2}+1} - 2b(1-\mu) + \\ +\left(\frac{1}{\mu}\right)^{\frac{1}{s_1}} \frac{(b-d)(1-\mu)^{\frac{1}{s_1}+1}}{\frac{1}{s_1}+1} - \left(\frac{1}{\mu}\right)^{\frac{1}{s_2}} \frac{(e-b)(1-\mu)^{\frac{1}{s_2}+1}}{\frac{1}{s_2}+1} \end{array}\right\} \tag{17}$$

Note: For $p_1 = p_2 = q_1 = q_2 = r_1 = r_2 = s_1 = s_2 = n$ we have the defuzzification value as (6),

$$D = \frac{1}{2}\left\{(a_1+c_1)\lambda + (2b-a_1-c_1)\frac{\lambda n}{1+n} - (a+c)\omega + (a+c-2b))\frac{\omega n}{1+n} + 2b(\mu-\delta) + \left(\frac{1}{\delta}\right)^{\frac{1}{n}} \frac{(e_1+d_1-2b)(1-\delta)^{\frac{1+n}{n}}}{\frac{n+1}{n}} + \left(\frac{1}{\mu}\right)^{\frac{1}{n}} \frac{(2b-d-e)(1-\mu)^{\frac{n+1}{n}}}{\frac{n+1}{n}}\right\} \tag{18}$$

Note: For the particular form $n = 2$, we have (7):

$$D = \frac{1}{2}\left\{(a_1+c_1)\lambda + (2b-a_1-c_1)\frac{2\lambda}{3} - (a+c)\omega + (a+c-2b))\frac{2\omega}{3} + 2b(\mu-\delta) + \left(\frac{1}{\delta}\right)^{\frac{1}{2}} \frac{(e_1+d_1-2b)(1-\delta)^{\frac{3}{2}}}{\frac{3}{2}} + \left(\frac{1}{\mu}\right)^{\frac{1}{2}} \frac{(2b-d-e)(1-\mu)^{\frac{3}{2}}}{\frac{3}{2}}\right\} \tag{19}$$

5 Model Development and Analysis

5.1 Model description and notations

We have extended Tsai & Wu [41] by considering investment as a crucial parameter to reduce the production of imperfect item. The paper considers an EPQ model for single item with constant demand in which the learning effect in the unit production time for an imperfect production system is investigated. The model we have observed that Q quantity of the item is produced by the manufacturer. The production process is not perfect and hence $\beta\%$ of defective items is produced. Thus $Q(1-\beta)$ are perfect quality items which are produced during regular production time T_1. Defective items are reworked after the regular production processes finishes and no extra defective items are produced. The production time is not fixed, and it depends on the learning effect of the worker because as the time goes on the worker become more experienced and hence requires less time to produce. Also, manufacturer invests money to reduce the imperfect production. The figure 6 and figure 7 given below depicts the on-hand inventory of non-defective and defective items.

In addition to the facts stated above, we assume the following here after:

- A constant setup cost is allotted for each cycle.
- Here, production rate of perfect quality items larger than the demand rate i.e., $Q(1-\beta) \geq DT_1$. Moreover, the rework rate is also higher than the demand rate.
- The manufacturer invests money to enhance the production process quality by buying new machinery, improving old machines through maintenance and repair, worker training, etc. We consider Porteus [43] logarithmic investment function as $I(\beta) = \frac{1}{\tau}\ln\left(\frac{\beta_0}{\beta}\right)$, where τ is the percentage decrease in β, per unit of currency increase in investment and β_0 is the original percentage of defective items produced before investment.
- Wright's [42] invented the learning phenomenon by suggesting the relation between man hour involved to produce unit item and collective production which is written as $T_j = T_1 j^{-l}$, where T_j denotes time to produce j^{th} unit and l is thFe learning rate *(0 < l < 1).* In this paper we have assumed the time required to produce x items under learning during regular production runs is $t^1(x) = a^{1x^{l^1}}$ while to produce y items under learning during reworking production runs is $t^1(y) = a_2 y^{l^2}$ where $-1< l_1,l_2<0$ are learning coefficients and a_1, a_2 are time required to produce the first unit for each cycle for regular and rework production run respectively.

To develop the model some notations are used throughout the research which are defined as follows:

Q	Production lot size for each cycle (decision variable);
D	Demand rate per unit time;
T_1	Regular production time;
T_2	Rework time;
T_3	Depletion time;
T	Cycle time ($T = T_1 + T_2 + T_3$);
l_1	Learning coefficient related with regular production, $l_1 = \frac{\ln c_1}{\ln 2}$, where C_1 is Learning rate in regular production;
l_2	Learning coefficient related with reworking production, $l_1 = \frac{\ln c_2}{\ln 2}$, where C_2 is Learning rate in reworking production;
τ	Percentage decrease in defective items per unit currency increase in investment;
S	Setup cost for each cycle;
c_1	Manual labour cost for production per unit time (inspection cost is incorporated);
c_2	Refurbish cost of defective quality items per unit time;
h_1	Holding cost for each perfect item (i.e., useful item) per unit time;
h_2	Holding cost for each defective quality item which are reworked per unit time;

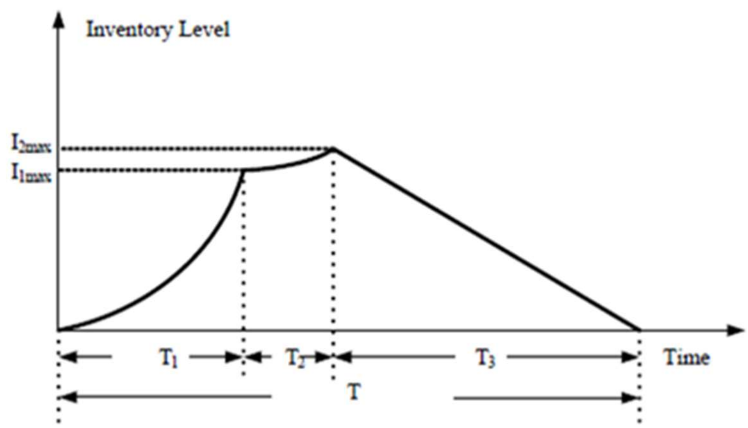

Figure 6.: On-hand inventory level of the non-defective items

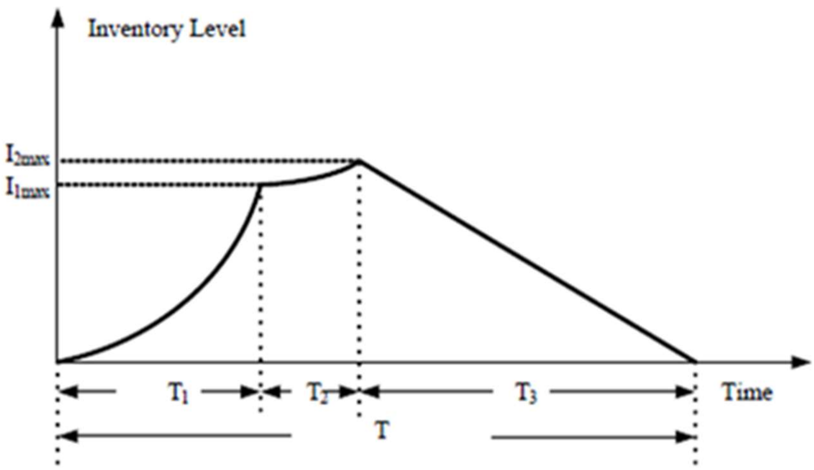

Figure 7.: On hand inventory level of the defective items

5.2 Learning in regular production

In this model we use Wright's Learning curve in production time of regular and reworked items in different way. Here we consider the learning rate in regular production runs is $t^1(x) = a^{1x^{l^1}}$ and that of during rework is $t^1(y) = a_2 y^{l^2}$ where $-1 < l^1, l^2 < 0$. Thus the total time taken for production of the Q regular items is given by (20):

$$T_1 = \int_0^Q a_1 x^{l_1}\, dx = \frac{a_1 Q^{l_1+1}}{l_1+1} \tag{20}$$

Thus, using $Q = \left[\frac{l_1+1}{a_1} T_1\right]^{\frac{1}{l_1+1}}$ which is the total quantity produced in regular production cycle (T_1).

Hence $Q_1(t) = \left[\frac{l_1+1}{a_1} t\right]^{\frac{1}{l_1+1}}$.

The total time taken to produce rework items is (9):

$$T_2 = \int_0^{\beta Q} a_2 y^{l_2}\, dy = \frac{a_2 (\beta Q)^{l_2+1}}{l_2+1} \tag{21}$$

Let us write, $Q_2(T_2) = \beta Q = \left[\frac{l_2+1}{a_2} T_2\right]^{\frac{1}{l_2+1}}$, (10):

Thus

$$Q_2(t) = \left[\frac{l_2+1}{a_2} t\right]^{\frac{1}{l_2+1}}. \tag{22}$$

During T_3 the inventory depletes as there is no production in this period but there is a constant demand *D* from the customer, thus $T_3 = \frac{Q-D(T_1+T_2)}{D}$.

5.2.1 Set up cost:

To start and establish any business a cost is incurred in the inventory which includes place to set up (factory, house for rent, machinery etc.) cost of machinery (to produce and to maintain), skilled worker, etc. The retailer must incorporate a constant set up cost SC = S.

5.2.2 Holding cost for the non-defective item:

During production, the perfect quality items are to be held for T_1, T_2 and T_3 period as shown in Fig 3.1. It is observed that the inventory increases during T_1 and T_2 and decreases during T_3. This is obvious because items are produced during regular production time T_1 and imperfect items are reworked to a perfect quality item during T_2, thus the inventory increases. While during T_3 the items get consumed due to the demand and there is no production during this period, hence the inventory decreases. Also, the inventory during T_3 is of perfect quality items. Thus, the holding cost of the non-defective items is (23):

$$HC_p = h_1\left(\int_0^{T_1}\left((1-\beta)Q_1(t) - Dt\right)dt + \int_0^{T_2}\left((1-\beta)Q - DT_1 + Q_2(t) - Dt\right)dt + \int_0^{T_3}(Dt)dt\right) = h_1\left(\frac{Q^2}{2D} + a_1Q^{l_1+2}\left(\frac{1-\beta}{l_1+2} - \frac{1}{l_1+1}\right) - \frac{a_2(\beta Q)^{l_2+2}}{(l_2+2)(l_2+1)}\right) \tag{23}$$

5.2.3 Holding cost of the defective items:

The inventory must hold the defective items during T_1 and T_2. During T_1 the defective items are produced with no rework and hence its inventory increases till T_1 where the maximum number of the defective items is βQ. After T_1 the rework starts and hence the inventory of the defective items decreases. Thus, the holding cost of defective items is (24):

$$HC_I = h_2\left(\int_0^{T_1}\beta Q_1(t)\,dt + \int_0^{T_2}\left(\beta Q - Q_2(t)\right)dt\right) = h_2\left(\frac{a_1\beta Q^{l_1+2}}{l_1+2} + \frac{a_2(\beta Q)^{l_2+2}}{(l_2+1)(l_2+2)}\right) \tag{24}$$

5.2.4 Production Cost:

Since the labour cost is considered per unit time and the items are produced till T_1, thus the labour cost is only cost of production which is (25):

$$PC = c_1T_1 = c_1\frac{a_1Q^{l_1+1}}{l_1+1} \tag{25}$$

5.2.5 Rework Cost:

As the defective items get reworked during T_2, thus rework cost (26):

$$RC = c_2T_2 = c_2\frac{a_2(\beta Q)^{l_2+1}}{l_2+1} \tag{26}$$

5.2.6 Investment Cost:

In our model the production quality is a major tool for the decision maker. Its management is required to lower the allied cost incurred for the manufacturing of better-quality item. Thus, is it is appropriate for the manufacturer to make investment and diminish the number of defective items produced. Assuming the logarithmic investment function, the cost incurred is IC (27):

$$IC = \frac{\vartheta}{\tau}\ln\left(\frac{\beta_0}{\beta}\right) \tag{27}$$

Where ϑ is the factional opportunity cost and $0<\beta<\beta_0<1$.

5.2.7 Total Cost (TC)

By adding up all the cost i.e., holding cost for non-defective and defective items, set up cost, production cost, reworking cost, we obtain the total cost. Thus, the average total cost i.e., total cost per unit time ($T=T_1+T_2+T_3=(Q/D)$) is given by $TC(Q,\beta)$ (28):

$$TC(Q,\beta) = \frac{1}{T}(SC + HC_P + HC_I + PC + RC + IC)$$

$$= S\frac{D}{Q} + h_1\left(\frac{Q}{2} + a_1Q^{l_1+1}D\left(\frac{1-\beta}{l_1+2} - \frac{1}{l_1+1}\right) - \frac{a_2D\beta^{l_2+2}Q^{l_2+1}}{(l_2+2)(l_2+1)}\right) + c_1\frac{a_1Q^{l_1}D}{l_1+1} + c_2\frac{a_2D\beta^{l_2+1}Q^{l_2}}{(l_2+1)} + h_2D\left(\frac{a_1\beta Q^{l_1+1}}{l_1+2} + \frac{a_2\beta^{l_2+2}Q^{l_2+1}}{(l_2+1)(l_2+2)}\right) + \frac{\vartheta Q}{\tau D}\ln\left(\frac{\beta_0}{\beta}\right) \tag{28}$$

Since Q is discrete and β is continuous so it is not possible to prove analytically that the total cost $TC(Q,\beta)$ is jointly convex. Hence, we check the optimality separately and obtain some condition under which the total cost is optimal.

For $-1 < l_1, l_2 \le 0$, let us consider $l_1 = -k_1,\ l_2 = -k_2, where\ 0 \le k_1,\ k_2 \le 1$ then for a given value of Q, $\frac{\partial^2 TC}{\partial \beta^2} = a_2 D\beta^{-k_2} Q^{1-k_2}\left(h_2 - h_1 - \frac{c_2 k_2}{\beta Q}\right) + \frac{\vartheta D}{\tau Q \beta^2} > 0, provided\ h_2 - h_1 - \frac{c_2 k_2}{\beta Q} > 0$. Thus $h_2 > h_1 + \frac{c_2 k_2}{\beta Q}$, hence we can also conclude that $h_2 > h_1$.

Let us relax the integer constrain of Q and let us consider it as real variable then we have proposed a lemma for optimality condition for the decision variable Q.

Lemma:

For $h_2 > h_1$ the total cost function $TC(Q,\beta)$ is strictly convex with respect to Q for all $Q < \min(X_1, X_2)$, where (29):

$$(X_1, X_2) = \left(\frac{c_1(k_1+1)(2-k_1)}{\beta(1-k_1)(h_2-h_1)}, \frac{c_2(k_2+1)(2-k_2)}{\beta(1-k_2)(h_2-h_1)}\right) \tag{29}$$

Proof: Differentiating the total cost $TC(Q,\beta)$ twice with respect to Q and rearranging the terms we get (30),

$$\frac{\partial^2 TC}{\partial Q^2} = \frac{2D}{Q^3}\left(S + \frac{\vartheta}{\tau} ln\left(\frac{\beta_0}{\beta}\right)\right) + \frac{a_1 k_1 D}{Q^{k_1+2}}\left(\frac{-\beta Q(1-k_1)}{(2-k_1)}(h_2 - h_1) + \frac{c_1(k_1+1)}{(1-k_1)}\right)$$
$$+ \frac{a_2 k_2 D \beta^{1-k_2}}{Q^{k_2+2}}\left(\frac{-\beta Q}{(2-k_2)}(h_2 - h_1) + \frac{c_2(k_2+1)}{(1-k_2)}\right) \tag{30}$$

In the above equation, the terms in the 2nd bracket is positive if (31)

$$Q < \frac{c_1(k_1+1)(2-k_1)}{\beta(1-k_1)(h_2-h_1)} \tag{31}$$

and the terms in the 3rd bracket is positive if

$$Q < \frac{c_2(k_2+1)(2-k_2)}{\beta(1-k_2)(h_2-h_1)} \tag{32}$$

Therefore $\frac{\partial^2 TC}{\partial Q^2} > 0$, i.e., the total cost $TC(Q,\beta)$ is strictly convex for all $Q < \min(X_1, X_2)\ and\ h_2 > h_1$.

We also have discussed an algorithm to optimize the decision variables in consecutive cycles where the production quantity may vary in all cycle. Here we consider the learning during production and rework as NLIVIFN and then proposed intuification results are applied. Thus, we get (33, 34),

$$\widetilde{l_1} = \frac{1}{2(n+1)}\left((l_{11} + l_{13})\lambda - (l_{14} + l_{15})\omega + 2l_{12}n(\lambda - \omega + \mu - \delta) + 2l_{12}(\mu - \delta) + n(1 - \delta)\left(\frac{1-\delta}{\delta}\right)^{\frac{1}{n}}(l_{16} + l_{17} - 2l_{12}) + n(1-\mu)\left(\frac{1-\mu}{\mu}\right)^{\frac{1}{n}}(2l_{12} - l_{18} - l_{19})\right) \tag{33}$$

$$\widetilde{l_2} = \frac{1}{2(n+1)}\left((l_{21} + l_{23})\lambda - (l_{24} + l_{25})\omega + 2l_{22}n(\lambda - \omega + \mu - \delta) + 2l_{22}(\mu - \delta) + n(1 - \delta)\left(\frac{1-\delta}{\delta}\right)^{\frac{1}{n}}(l_{26} + l_{27} - 2l_{22}) + n(1-\mu)\left(\frac{1-\mu}{\mu}\right)^{\frac{1}{n}}(2l_{22} - l_{28} - l_{29})\right) \tag{34}$$

Then (35):

$$\widetilde{TC}(Q,\beta) == S\frac{D}{Q} + h_1\left(\frac{Q}{2} + a_1 Q^{\widetilde{l_1}+1} D\left(\frac{1-\beta}{\widetilde{l_1}+2} - \frac{1}{\widetilde{l_1}+1}\right) - \frac{a_2 D \beta^{\widetilde{l_2}+2} Q^{\widetilde{l_2}+1}}{(\widetilde{l_2}+2)(\widetilde{l_2}+1)}\right) + c_1 \frac{a_1 Q^{\widetilde{l_1}} D}{\widetilde{l_1}+1} + c_2 \frac{a_2 D \beta^{\widetilde{l_2}+1} Q^{\widetilde{l_2}}}{(\widetilde{l_2}+1)} +$$

$$h_2 D\left(\frac{a_1 \beta Q^{\widetilde{l_1}+1}}{\widetilde{l_1}+2} + \frac{a_2 \beta^{\widetilde{l_2}+2} Q^{\widetilde{l_2}+1}}{(\widetilde{l_2}+1)(\widetilde{l_2}+2)}\right) + \frac{\vartheta Q}{\tau D} \ln\left(\frac{\beta_0}{\beta}\right) \quad (35)$$

It is observed that the control parameters are not independent of each other. So in order to obtain the optimum solution let us acclimatize an iterative algorithm as below.

5.3 Algorithm:

To obtain the optimum total cost with respect to the decision variables we must follow an algorithm. Thus, in this section an algorithm is discussed step by step to understand the process.

- Step 1: Set *i = 1* and $\beta = \beta_0$
- Step 2: Solve for Q by $\frac{\partial TC}{\partial Q} = 0$ and obtain $Q = Q_0$
- Step 3: From the above value of $Q = Q_0$ obtain the value of β by using $\frac{\partial TC}{\partial \beta} = 0$ as $\beta = \beta^*$
- Step 4: Again, substitute the value $\beta = \beta^*$ in step 2 and obtain the optimum value of Q as $Q = Q^*$
- Step 5: Evaluate the optimal average total cost $\widetilde{TC}^*(Q^*, \beta^*)$ from ( ) using $\beta = \beta^*$ *and* $Q = Q^*$
- Step 6: Set *i = i + 1* and considering $a_{1(i+1)} = a_{1i} Q^{l_1}$ *and* $a_{2(i+1)} = a_{2i} Q^{l_2}$
- Step 7: If $|Q_i^* - Q_{i-1}^*| > \epsilon$ then go to step 2 else $i^* = i - 1$ and stop.
  Thus i* is the number of cycles till we have the maximum learning occurs.

5.4 Flowchart:

The step-by-step algorithm is described by a chart to understand the flow of the problem and the way to optimize the model. Thus, the flowchart is given below (Figure 8.):

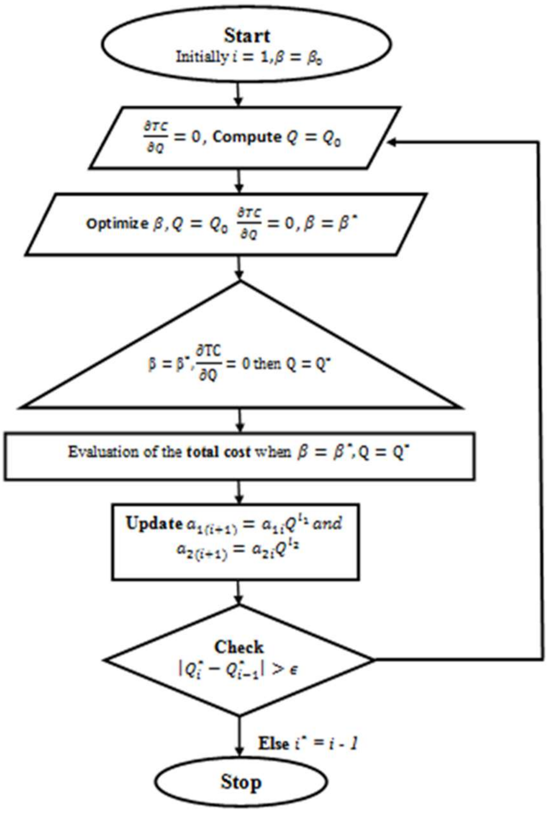

Figure 8.: Flowchart for the problem

5.5 Numerical study

A numerical experiment with the following input parameter is conducted whose results are reported further. Also, the learning during the production is 95% (which is equivalent to $l_{12} = -0.089$) while that during rework is 91% (which is equivalent to $l_{22} = -0.136$).

Parameters:

$$r = 60; S = 20000; \mathrm{h}_1 = 8; \mathrm{h}_2 = 20; \mathrm{c}_1 = 1000; \mathrm{c}_2 = 400; \beta_0 = 0.2; \mathrm{a}_1 = 0.01; \mathrm{a}_2 = 0.008;$$
$$\mathrm{C}_1 = 0.94; \mathrm{C}_2 = 0.91; \vartheta = 0.02; \tau = 0.0002; l_{12} = -0.089; l_{22} = -0.136; l_{11} = -0.093;$$
$$l_{13} = -0.085; l_{14} = -0.091; l_{15} = -0.087; l_{16} = -0.095; l_{17} = -0.083; l_{18} = -0.092;$$
$$l_{19} = -0.086; l_{21} = -0.11; b_{23} = -0.17; l_{24} = -0.13; l_{25} = -0.15; l_{26} = -0.10;$$
$$l_{27} = -0.18; l_{28} = -0.12; l_{29} = -0.16; \delta = 0.1; \mu = 0.2; \omega = 0.8; \lambda = 0.9; n = 2;$$

Thus, if we compare our nonlinear model with Tsai model, we can observe that, investment of money for machine modernization or upgradation reduces the production of defective items with increase in produced quantity. It is also observed that the optimum quantity (Q*) produced in case of linear and non-linear IVFN environment is more as compared to the Tsai model and classical model, which means that model works better in uncertain environment which is desirable. Considering the learning effect by investing in the model increases the production of the items in both uncertain and crisps environment. Also, due to the increase in investment, there is a decrease in the production of defective items which results in good reputation and a decrease in total cost as compared to the previously established model.

Table 2: Comparative result of the numerical example.

	Q*	β*	TC^*	T_1^*	T_2^*	T_3^*	T^*
Our Model (in Non - Linear IVFN n=2)	636	0.0087	4018.21	3.93	0.0406	6.629	10.6
Our model (in Linear IVFN taking n=1)	715	0.0048	3773.08	4.373	0.0269	7.517	11.92
Our Model (in Crisps)	690	0.0062	3845.21	4.227	0.032	7.24	11.5
Tsai and Wu model (τ=0)	455	0.2	5532.11	2.89	0.46	4.234	7.583
Classical model (l_1=l_2=0, β=0,τ=0)	548	0	4981.78	5.48	0	3.653	9.133

Figure 9 and figure 10 observe the comparative study of different models with respect to optimum quantity and optimum total cost respectively.

Tsai [41] has considered constant defective item, which is not realistic, thus we have developed our model by considering defective item as variable quantity. In our proposed model we have optimized not only the quantity Q but also the number of defective items.

From Table 3 we observe that the learning is maximum in 10th cycle for the crisps environment.

Table 3: Optimal solution till learning occurs in crisps arena

Cycle	Q*	β*	T_1^*	T_2^*	T_3^*	T^*	Total Cost (TC^*)
1	690	0.0062	4.227	0.032	7.24	11.5	3845.21
2	614	0.0129	2.121	0.023	8.09	10.23	4118.78
3	581	0.0249	1.137	0.016	8.53	9.683	4250.03

Cycle	Q*	β*	T_1^*	T_2^*	T_3^*	T^*	Total Cost (TC^*)
4	565	0.0458	0.628	0.011	8.778	9.417	4316.76
5	557	0.0824	0.352	0.008	8.923	9.283	4350.64
6	552	0.1461	0.198	0.005	8.996	9.2	4366.85
7	549	0.2571	0.112	0.004	9.034	9.15	4373.32
8	547	0.4492	0.064	0.002	9.05	9.117	4374.32
9	546	0.7818	0.036	0.002	9.062	9.1	4372.26
10	546	0.9801	0.021	0.0009	9.078	9.1	4368.88

From Table 4 we can observe that the maximum learning occurs in 6^{th} cycle for non-linear IVFN environment.

Table 4: Optimal solution till learning occurs in non-linear IVFN arena

Cycle	Q*	β*	T_1^*	T_2^*	T_3^*	T^*	Total Cost (TC^*)
1	636	0.0087	3.93	0.0406	6.629	10.6	4018.21
2	574	0.025	1.478	0.016	7.956	9.567	4274.57
3	556	0.064	0.6137	0.0062	8.613	9.267	4351.91
4	550	0.156	0.2593	0.0024	8.905	9.167	4372.77
5	547	0.354	0.1099	0.0008	9.006	9.133	4374.11
6	547	0.41	0.0468	0.0002	9.086	9.133	4373.67

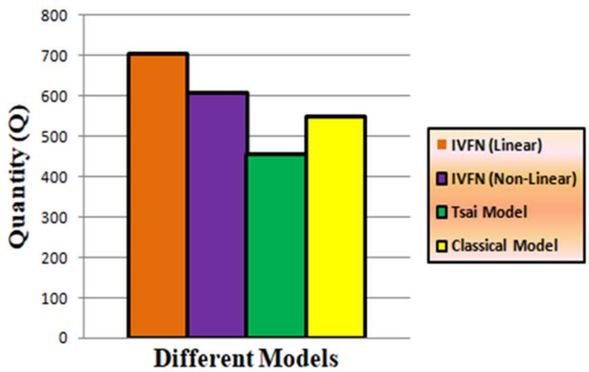

Figure 9: Changes in different model w.r.t Q

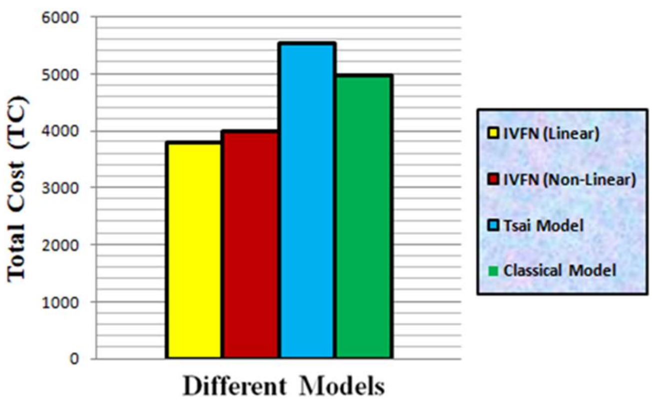

Figure 10: Changes in different model w.r.t TC

Figure 11 shows the comparitive study of the optimum quantity in crisps and NLIVIFN environment, while figure 12 shows the comparitive study of the optimum total cost in crisps and NLIVIFN environment.

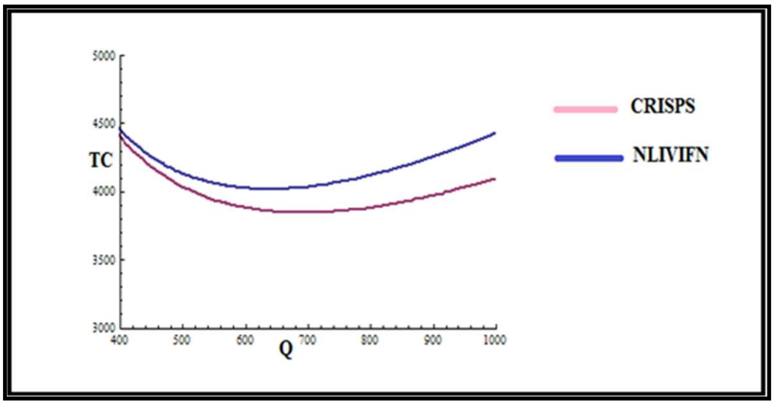

Figure 11: Comparison between Crisps and NLIVIFN w.r.t. Q

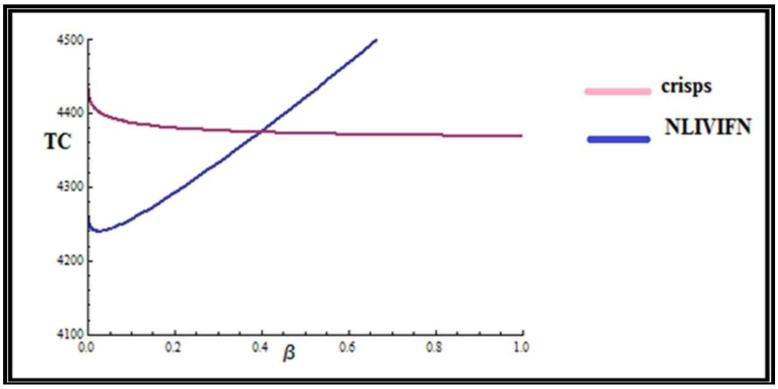

Figure 12: Comparison between Crisps and NLIVIFN w.r.t. β

Discussion: From the figure 11 it is observed that, if we consider our model under NLIVIFN than the total cost will increase if the production of defective items increases. Also the defective items produced are at maximum 45% in NLIVIFN while that in crisps environment percentage of defective items increases more with steady cost. From figure 12, quantity produced is more in NLIVIFN as that of crisps environment.

If we compare table 3 and table 4 we can see that it requires 10 cycles to obtain learning effect under crisps environment while in just 6 cycles to obtain the same in NLIVIFN arena. Again more quantity of perfect quality items and less number of defective items are produced under NLIVIFN arena than under crisps arena. It is also observed that it take less cycle time (T) in NLIVIFN. Thus the proposed model proposes to consider learning as NLIVIFN.

Figure 13 and Figure 14 shows the 2D plot of total cost with respective to the percentage of defective items (*β%)* and optimum quantity (Q) respectively, while figure 15 shows 3D plot of total cost with respect to the above two variables. Thus, the detail comparative study of our crisps, linear and NLIVIFN with the classical and Tsai model is done below in table 5.

Table 5: Effect of parameter β_0

Parameters	Values	Quantity (Q^*)	Defective fraction (β^*)	Total Cost (TC^*)	$IC(\beta^*)$
β_0	0.005	632	0.005	3984.69	0.00
	0.01	632	0.008	3989.93	22.31
	0.1	635	0.0084	4011.68	247.69
	0.2	636	0.0087	4018.21	313.5
	0.4	637	0.0094	4024.74	375.07
	0.68	637	0.011	4029.92	412.42

An essential insight is derived from Table 5 that the investment made to improve the product quality is dependent on the original quality items. This is obvious, because for very low percentage of defective item (β_0) in original manufacturing process, there is no requirement by the retailer to invest money to improve the product quality. It is also observed that if the original defective percentage is more, than the investment is required to improve the process quality also increases. Also, due to investment there is a decrease in production of 98% of defective items.

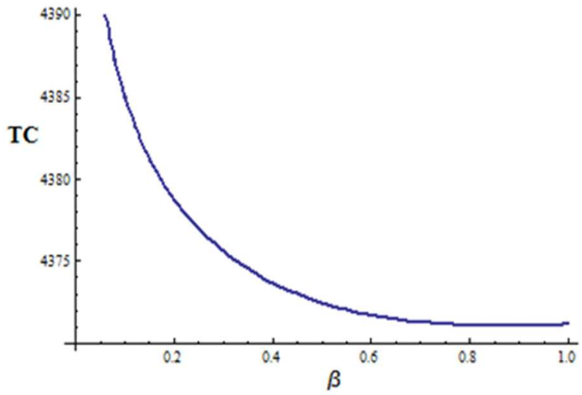

Figure 13: The behaviour of TC w.r.t. β

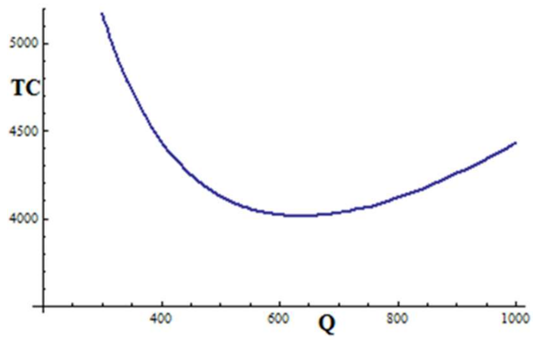

Figure 14: The behaviour of TC w.r.t. Q

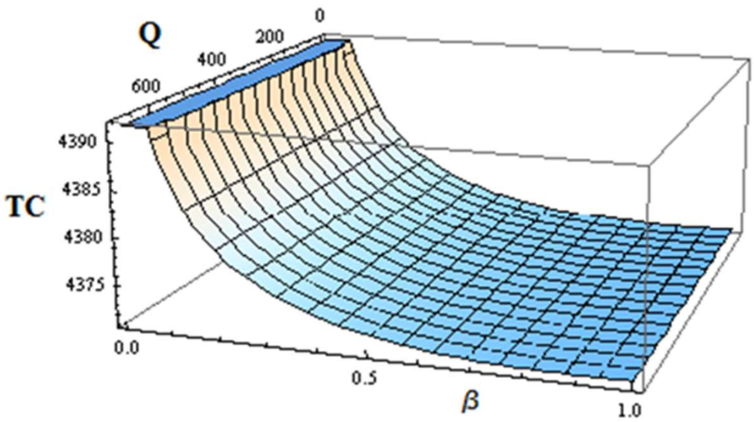

Figure 15: The behaviour of TC w.r.t. Q and β

6 Sensitivity Analysis

Let us observe the sensitivity of the various parameters considering learning as NLIVIFN is done below in Table 6.

Table 6: Sensitivity Analysis of various parameters

Parameters	Values	Quantity (Q^*)	Defective fraction (β^*)	Total Cost (TC^*)	$IC(\beta^*)$
D	40	495	0.0132	3421.64	271.81
	50	567	0.01	3748.29	299.57
	70	704	0.0074	4242.21	329.68
	80	772	0.0064	4427.34	344.20
h_1	4	897	0.004	2935.12	391.20
	6	734	0.0061	3525.37	349.00
	10	570	0.0120	4449.05	281.34
	12	520	0.0170	4835.83	246.51
a_1	0.008	616	0.011	4102.73	290.04
	0.009	626	0.0097	4062.02	302.62
	0.011	647	0.0078	3973.28	324.42
	0.012	658	0.0071	3927.2	333.82
S	10000	457	0.0157	2921.7	254.47
	15000	555	0.0111	3514.4	289.14
	25000	708	0.0072	4464.62	332.42
	30000	773	0.0062	4869.71	347.38
τ	0.00015	637	0.0116	4027.46	379.64
	0.00025	636	0.0069	4012.16	268.19
	0.00030	635	0.0057	4007.87	237.19

6.1 *Discussion:*

From the above table 6 it is observed that with the increase in demand rate (D), time required to produce the first unit (a_1) in production process, Set up Cost (S), the production lot size increases and the percentage of the defective items decreases due to the learning effect and additional investment. This is obvious because with the additional investment in the inventory, a larger production of lot sizes occurs. At the same time, the quality of production improves due to the investment, resulting in fewer defective items over time. Also, increasing the set-up cost, improves the maintenance of the items. This leads to increase in total costing as well as increase in investment cost for reduction in production of defective items. While it is observed that with the increase in holding cost of the perfect items (h_1) the produced quantity decreases and the percentage of the defective items increases. This is realistic because if the holding per unit cost increases, then the manufacturer should produce less in order to optimize the total cost. As, τ is inversely proportional to the investment so with the increase in the τ, means decrease in investment cost. Also, as τ increases, the total cost and the percentage of the defective item decreases.

6.2 *Advantages:*

The model works best in uncertain and crisps environment as compared to the existing models. As the model considers number of defective item as variable, so, it is realistic as the production of defective items can't be determine from the prior and it is affected by the learning behaviour of the worker. Thus, in this model the effect of learning is also incorporated so that due to learning, the production of defective items decreases.

6.3 Limitations:

This model considers in non-linear interval valued IFN whereas it can be also done in neutrosophic environment. The model considers the effect of learning and investment in the production non-defective items. The other aspect of the inventory model can be incorporated in future work.

7 Conclusion

In this article, we initiated the concept of non-linear interval-valued intuitionistic fuzzy set and their graphical classifications are discussed here. Also, a new de-intuitification method is developed in NLIVIFN arena using alpha/beta cut method. Further, we proposed an operation research related problem in inventory field and discussed the optimal results. In the example of EPQ model we have observed that the while taking learning as NLIVIFN then the maximum learning occurs faster as compare to crisps. Additionally, we have observed that due to the worker's experience the time taken to produce the item and to rework the defective item decreases as the learning increases till saturation. Also, the optimal quantity produced is more and production of defective item is less in NLIVIFN environment. Sensitivity analysis is also done here to justify the results and usefulness of de-intuitification is also discussed in this article. Lastly, we observe that de-intuitification results have a great impact in uncertainty based inventory control problem.

In future, researchers can incorporate lots of interesting algorithms using NLIVIFN in different fields like medical diagnosis, pattern recognition, image processing, big-data analysis, economic and social-science based on different real-life problems, etc.

Conflict of Interest:

The authors have no conflict of interest in this research article.

Data Availability:

No real-life data is associated with this research article.

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